Suppose that \( A \) is a \( 4 \times 4 \) matrix with columns that are equal to the vectors
\[
\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}, \quad \text{and} \quad \vec{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}.
\]

**Problem 6.1.** Explain why the fact that \( A \) is invertible is equivalent to . . .

(a) The columns of \( A \) are linearly independent.

If matrix is invertible, it means that it can be multiplied by another matrix to produce an identity matrix. Since the matrix is an \( n \times n \) matrix, there are chances of [the columns] being linearly independent [the matrix being invertible and therefore have linearly independent columns]. In order for the matrix to be invertible the matrix must be able to be reduced to an identity matrix which is how you find [if the columns of] a matrix [are] linearly independent. Also an a matrix with an inverse must contain columns that span the vector space, in this case \( \mathbb{R}^4 \). If one of the vectors is linearly dependent that the vectors cannot span the entire vector space. If any of these are untrue, the matrix does not have in inverse.

We never say a matrix is linearly independent. But the columns of a matrix can be linearly independent. The matrix does not have to be square to have linearly independent columns.

If \( A \) is invertible, there exists a matrix \( A^{-1} \) such that \( A^{-1}A = I = AA^{-1} \). Linearly independent means that the only solution to \( \vec{a}x_1 + \vec{b}x_2 + \vec{c}x_3 + \vec{d}x_4 = \vec{0} \) is trivial solution. Therefore, we can put the previous equation into matrix vector form \( AX = \vec{0} \), where \( A \) is augmentation of all 4 vectors. We can multiply by \( A^{-1} \) on both sides, \( A^{-1}AX = A^{-1}\vec{0} \), and, by definition on invertibility, \( I\vec{x} = A^{-1}\vec{0} \), and since anything by the identity matrix is itself, and anything by the zero vector is the zero vector, it reduces down to \( \vec{x} = \vec{0} \), proving the definition of linear independence.

(b) The only solution of the equation \( AX = \vec{0} \) is \( \vec{x} = \vec{0} \).

If the only solution of \( AX = \vec{0} \) is \( \vec{x} = \vec{0} \), it means that the 4 vectors are linearly independent.

\[
[\vec{a}, \vec{b}, \vec{c}, \vec{d}]\vec{x} = \vec{0}
\]

\[
(A^{-1})AX = (A^{-1})\vec{0}
\]
\[ I\vec{x} = \vec{0} \]
\[ \vec{x} = \vec{0} \]

Invertible means that there is a matrix that can be multiplied by the original matrix to get the identity matrix. And as shown by the above, the invertibility of the matrix proves that the only solution of to \( A\vec{x} = \vec{0} \) is \( \vec{x} = \vec{0} \) and in turn that calls the theorem that invertible matrices’ columns are linearly independent, which again refers to the solution set of \( A\vec{x} = \vec{0} \) being \( \vec{x} = \vec{0} \), a.k.a. trivial.

A definition of \( A \) is invertible includes there is some \( C \) so that \( CA = I_n = AC \). If that is true, we can set \( [\vec{a}, \vec{b}, \vec{c}, \vec{d}] = A \). Then \( \vec{x} = I_n\vec{x} = CA\vec{x} = C\vec{0}(since \ A\vec{x} = \vec{0}) = \vec{0} \) so only solution to \( A\vec{x} = \vec{0} \) is the trivial one.

(c) When put in reduced echelon form, \( A \) becomes \( I_4 \).

A matrix is invertible if \( A^{-1}A = I_4 \) and \( AA^{-1} = I_4 \). If
\[
\begin{bmatrix}
a_1 & b_1 & c_1 & d_1 \\
a_2 & b_2 & c_2 & d_2 \\
a_3 & b_3 & c_3 & d_3 \\
a_4 & b_4 & c_4 & d_4
\end{bmatrix}
\]
is reduced to
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
then the matrix is invertible. Putting \( A \) into reduced echelon form is the method used to find \( A^{-1} \). If \( A^{-1} \)
exists then the matrix is invertible by definition.

(d) When put in echelon form, \( A \) has \( 4 \) pivot positions.

Theorem.

(1) A is invertible if multiplied by its inverse \( A^{-1} \), it produced identity matrix: \( A^{-1}A = I_n = AA^{-1} \)

(2) Also \( A \) is invertible if it is row equivalent to the identity matrix in row reduced echelon form

Connection: According to (2), \( A \) is invertible if it is row equivalent to the identity. Since the identity is an \( n \times n \) matrix with \( n \)-pivot (definition), \( A \) must have 4 pivot positions to be invertible.

If \( A \) is invertible there exists matrix \( C \) such that \( CA = AC = I_4 \). If \( A \) has 4 pivot positions when put in echelon form, that means that there is 1 unique solution to the \( 4 \times 4 \) matrix [the system \( A\vec{x} = \vec{w} \) for all vectors, \( \vec{w} \)]. These statements are equivalent b/c if \( A \) can be manipulated to the \( I_4 \) matrix [which is how you find the inverse], that is the same as having 4 pivot positions. The below matrices are equivalent:

\[
\begin{bmatrix}
1 & \# & \# & \# \\
0 & 1 & \# & \# \\
0 & 0 & 1 & \# \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

(e) For each \( \vec{w} \in \mathbb{R}^4 \) there is a solution of the equation \( A\vec{x} = \vec{w} \).

If \( A \) is invertible then there exists a matrix \( C \) such that \( CA = I_n = AC, C \) is the same as \( A^{-1} \). If there is a solution for each \( \vec{w} \in \mathbb{R}^4 \), this means that the columns of \( A \) span \( \mathbb{R}^4 \) and that there is a solution for \( x \), which gives a solution to \( A\vec{x} = \vec{w} \) for each \( \vec{w} \in \mathbb{R}^4 \). This is true if \( A \) is invertible because for the equation \( A\vec{x} = \vec{w} \):

\[ A^{-1}A\vec{x} = A^{-1}\vec{w} \]
\[ I_n\vec{x} = A^{-1}\vec{w} \]
\[ \vec{x} = A^{-1}\vec{w} \]

If \( A \) is invertible, there exists a solution \( \vec{x} = A^{-1}\vec{w} \) for each \( \vec{w} \in \mathbb{R}^4 \).

(f) The columns of \( A \) span \( \mathbb{R}^4 \).

Write your own solution to this. It may be helpful to use the work from part e.
Problem 6.2. Considering the matrix $A$ described above 6.1, suppose the span of the columns of $A$ is dimension 3. (Another way to say this is $\text{rk}(A) = 3$.) Which of the following are consequences? Circle all that apply.

There are frequently several ways to reason through multiple choice questions. Below is one way to think about it.

- The columns of $A$ span $\mathbb{R}^4$.
  
  **FALSE** *This is false because the space spanned by the columns has dimension 3 and the dimension of $\mathbb{R}^4$ is 4.*

- $A$ is not invertible.
  
  **TRUE** *If $A$ were invertible, then by the Invertible Matrix Theorem the columns would span $\mathbb{R}^4$, which they do not.*

- The columns of $A$ are linearly dependent.
  
  **TRUE** *If the columns of $A$ were linearly independent, then by the Invertible Matrix Theorem, $A$ would be invertible.*

- The solution set to $A\vec{x} = \vec{0}$ is only the trivial solution, $\{\vec{0}\}$.
  
  **FALSE** *Since the columns of $A$ are linearly dependent, the system
  
  $x_1\vec{a} + x_2\vec{b} + x_3\vec{c} + x_4\vec{d} = \vec{0}$
  
  has a nontrivial solution. The vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is a nontrivial solution to $A\vec{x} = \vec{0}$. Another way to argue this is directly from the Invertible Matrix Theorem.*

- $\det(A) = 0$.
  
  **TRUE** *Since $A$ is not invertible, $\det(A) = 0$.*

- For every choice of $\vec{w} \in \mathbb{R}^4$, the system $A\vec{x} = \vec{w}$ has infinitely many solutions.
  
  **FALSE** *It is entirely possible that this system is inconsistent. (Try coming up with an example.)*