Week #6: More on Matrix Properties

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http://math.ucsb.edu/~kgracekennedy/Spring2013_4A.html

6.1 Matrix Operations

Problem 6.1 (2.1.4). Compute $A - 5I_3$ and $(5I_3)A$, where

$$A = \begin{bmatrix} 5 & -1 & 3 \\ -4 & 3 & -6 \\ -3 & 1 & 2 \end{bmatrix}$$

Problem 6.2 (2.1.7). If a matrix is $5 \times 3$ and the product $AB$ is $5 \times 7$, what is the size of $B$?

Problem 6.3 (2.1.12). Let $A = \begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix}$. Construct a $2 \times 2$ matrix $B$ such that $AB$ is the zero matrix. Use two different nonzero columns for $B$.

Problem 6.4 (1.9.13). Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that $T(e_1)$ and $T(e_2)$ are the vectors shown in the figure. Using the figure, sketch the vector $T(2,1)$.

![Diagram showing T(e1) and T(e2)]
6.2 Invertible Matrices

Problem 6.5 (2.2.18). Solve the equation $AB = BC$ for $A$ assuming that $A$, $B$, and $C$ are square and $B$ is invertible.

Problem 6.6 (2.2.1). Find the inverse of the matrix

$$
\begin{bmatrix}
8 & 6 \\
5 & 4
\end{bmatrix}
$$

Problem 6.7 (2.2.5). Use the inverse found in Problem 6.6 to solve the system

\begin{align*}
8x_1 + 6x_2 &= 2 \\
5x_1 + 4x_2 &= -1
\end{align*}

Problem 6.8 (Test Question Last Quarter). There is a secret $2 \times 3$ matrix $C$ for which

$$
C \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \vec{e}_1 \quad C \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = \vec{e}_2
$$

Find a $D$ so that $CD = I_2$ and solve for the two vectors shown:

$$
D = \begin{bmatrix} \ \\
\end{bmatrix}, \quad C \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \vec{e}_2, \quad C \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \vec{e}_2
$$

Is it possible to find two matrices of the same size that when multiplied together will give you $I_3$?
Problem 6.9 (2.2.24). Suppose $A$ is $n \times n$ and the equation $Ax = b$ has a solution for each $b \in \mathbb{R}^n$. Explain why $A$ must be invertible. [Hint: Is $A$ row equivalent to $I_n$?]

Problem 6.10 (2.2.16). Suppose $A$ and $B$ are $n \times n$ matrices, $B$ is invertible, and $AB$ is invertible. Show $A$ is invertible. [Hint: Let $C = AB$.]

Problem 6.11 (2.1.23). Suppose $CA = I_n$ (the $n \times n$ identity matrix). Show that the equation $Ax = 0$ has only the trivial solution. Could $A$ have more columns than rows? Could $A$ have more rows than columns?
6.3 Determinants

Problem 6.12 (3.1.9). Compute by cofactor expansion the determinant of

\[
\begin{vmatrix}
6 & 0 & 0 & 5 \\
1 & 7 & 2 & -5 \\
2 & 0 & 0 & 0 \\
8 & 3 & 1 & 8
\end{vmatrix}.
\]

Problem 6.13 (3.2.35). Let \( U \) be a square matrix such that \( U^T U = I \). Show that \( \det U = \pm 1 \).

Problem 6.14 (3.2.32). Find a formula for \( \det(rA) \) where \( A \) is an \( n \times n \) matrix.

Problem 6.15 (Exam Question Last Quarter). Compute \( \det(AB^3A^{-1}) \) where

\[
A = \begin{bmatrix}
2 & 1 & 4 \\
1 & 0 & 3 \\
-2 & -3 & 4
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 & 1 \\
2 & 1 & 4 \\
1 & 2 & 1
\end{bmatrix}
\]

\[
\det(AB^3A^{-1}) =
\]
Problem 6.16 (3.2.5). Find by row reduction the determinant of

\[
\begin{vmatrix}
1 & 5 & -6 \\
-1 & -4 & 4 \\
-2 & -7 & 9
\end{vmatrix}.
\]

In other words, reduce this matrix to row echelon form and use the properties of how determinants transform under row operations.

Problem 6.17 (Exam Question Last Quarter). Jack tells Jill that he has a $3 \times 3$ matrix $A$ so that

\[ A = -A^{-1} \]

Jill says Jack’s wrong, there is no such matrix of real numbers. Who is right and why? (Hint: Find det $A$.)

Jill tells Jack that she has a $2 \times 2$ matrix $A$ so that $A = -A^{-1}$. Fill out the missing two entries:

\[
A = \begin{bmatrix}
\phantom{-}1/2 & -1/2 \\
\phantom{-}13/2 & \phantom{-}2
\end{bmatrix}
\]

If no such $A$ exists, say so here: