Math 5C Practice Final

1. Consider the following function:

\[ f(x) = x \quad x \in [0, 2\pi) \]

a) Find the Fourier series for this function and sketch its periodic extension.
b) Find the Fourier sine series for this function and sketch the periodic odd extension.
c) Find the Fourier cosine series for this function and sketch the periodic even extension.

2. Find the Fourier sine series for the following function:

\[ f(x) = x(1 - x) \quad x \in [0, 1] \]

3. Find the Fourier cosine series for the following function:

\[ f(x) = 3x^2 - 2x^3 \quad x \in [0, 1] \]

Hint: You already know the Fourier sine series for \( x(1-x) \).

4. Solve the eigenvalue problem: \( f(x) = \lambda f(x) \) for each of the following boundary conditions:

a) \( f(0) = f(L) = 0 \)
b) \( f'(0) = f'(L) = 0 \)
c) \( f(0) = f'(L) = 0 \)
d) \( f'(0) = f(L) = 0 \)
e) \( f(x) = f(x + 2\pi) \)

5. Use separation of variables to transform each P.D.E. into two O.D.E.’s.

a) \( c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \)

b) \( \frac{\partial^2 u}{\partial x^2} = p(t) \frac{\partial u}{\partial t} \)

c) \( c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \)
d) \[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]

e) \[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \]

6. Solve the following P.D.E’s. Use the results from previous problems where applicable.

a) \[ \frac{\partial^2 u}{\partial x^2} = (2t + 1) \frac{\partial u}{\partial t} \quad \frac{\partial u}{\partial x} (0,t) = \frac{\partial u}{\partial x} (1,t) = 0 \quad u(x,0) = 3x^2 - 2x^3 \]

b) \[ 16 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad u(0,t) = u(1,t) = 0 \quad u(x,0) = x(1-x) \]

\[ \frac{\partial u}{\partial t} (x,0) = 0 \]

c) \[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad u(r, \theta) = u(r, \theta + 2\pi) \]

\[ u(1, \theta) = 4 - \cos(\theta) + 3\sin(2\theta) + 5\cos(3\theta) \]

\[ u \text{ is well behaved at } r = 0. \]

d) \[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad u(x,0) = u(x,\pi) = u(y,0) = 0 \]

\[ u(\pi, y) = \sin(y) + 2\sin(2y) - 5\sin(4y) \]

e) \[ 100 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad u(0,t) = \frac{\partial u}{\partial x} (\pi,t) = 0 \]

\[ u(x,0) = \sin\left(\frac{x}{2}\right) - 2\sin\left(\frac{3x}{2}\right) \quad \frac{\partial u}{\partial t} (x,0) = 4\sin\left(\frac{3x}{2}\right) \]

f) \[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad u(r, \theta) = u(r, \theta + 2\pi) \quad u(1, \theta) = \theta \]

\[ u \text{ is well behaved at } r = 0. \]