Math 108A Midterm Examination II Name Solutions

November 10, 2009

For full credit include complete explanations of your answers and show all work for each problem.

(/6) 1. Suppose that V is finite dimensional and U is a subspace of V such that dim U = dim V. Prove that U = V.

Let $\{u_1, \dots, u_n\}$ be a basis for U. We must show that it spans V. Suppose not. Then there is a vector $v \notin U$ and $\{u_1, \dots, u_n, v\}$ is independent because if $a_1 u_1 + \cdots + a_n u_n + b v = 0$ then $b \neq 0$ implies that $v = \frac{a_1}{b} u_1 + \cdots + \frac{a_n}{b} u_n \in U$, a contradiction to $v \notin U$. If $b = 0$, $\{u_1, \dots, u_n\}$ independent implies $a_i = 0, i = 1, \dots, n$. So the set of $n+1$ vectors is independent. But this contradicts $\dim V = n$. So $\{v_1, \dots, u_n\}$ spans V.

(/6) 2. Suppose that T:V → W is a linear map. Prove that T is injective if and only if null T = {0}.

($\Rightarrow$) Suppose T is injective, then 0 is the only vector such that $T(0) = 0$ so null $T = \{0\}$.

($\Leftarrow$) Suppose null $T = \{0\}$ and there are distinct vectors v, v' with $T(v) = T(v')$, i.e. T is not injective. Then $T(v - v') = T(v) - T(v') = 0$. So $v - v' \in$ Null T giving $v - v' = 0$ or $v = v'$ contradicting v, v' distinct.

3. Let $\{v_1, v_2, v_3\}$ be a basis of a vector space, V, and $\{w_1, w_2, w_3, , w_4, w_5\}$ a basis of the vector space W. Let T:V → W be a linear map and M(T) be the matrix representing T as a matrix with respect to the basis.

(/6) (i) Suppose that $T(v_1) = 3w_1 - w_3 + w_5$, $T(v_2) = -w_1 + 2w_2 + w_4$, and $T(v_3) = -w_2 + 2w_3 - w_5$. What is M(T)?

$$M(T) = \left(T(v_1) \mid T(v_2) \mid T(v_3) \right) = \begin{pmatrix} 3 & -1 & 0 \\ 0 & 2 & -1 \\ -1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

(/6) (ii) Give an example of M(T) for which dim null T = 1.

$$M(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Math 108A Midterm Examination II Name Solutions
November 10, 2009

(/6) (iii) Give an example of M(T) for which dim rang T = 4.

This is impossible because

$3 = \dim V = \dim \operatorname{null} T + \dim \operatorname{ran} T$

so $\dim \operatorname{rang} T$ is never larger than 3.

(/6) (iv) What is dim $\mathcal{L}$(V,W)?

$\dim \mathcal{L}(V, W) = \dim \operatorname{Mat}(5, 3) = 15$

(/6) (v) If dim null T = 2, what is dim rang T?

$$3 = \dim V = \dim \operatorname{null} T + \dim \operatorname{rang} T$$
$$= 2 + \underline{1}$$

$\dim \operatorname{rang} T = 1$

(/6) 4. (i) Prove that every polynomial with odd degree and real coefficients has a real root.

Suppose that $p(x)$ is a real polynomial of odd degree, $2n+1$. By the fundamental theorem of algebra, $p(x)$ has $2n+1$ roots over the complex numbers. Complex (non-real) roots occur in conjugate pairs so the number of non-real roots is even. But $2n+1$ is odd so there must be at least one real root.

Math 108A Midterm Examination II
November 10, 2009

Name Solutions

(ii) What is the largest number of distinct real roots that a polynomial of degree 7 with real coefficients can have? Explain and give an example.

Suppose $p(x)$ is real of degree 7, then it can have at most seven (7) distinct real roots by the Fundamental theorem of algebra. For example we could have

$$p(x) = (x-1)(x-2)(x-3)(x-4)(x-5)(x-6)(x-7)$$

(/6) 5. Suppose that W is finite dimensional and $T \in \mathcal{L}(V,W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W,V)$ such that ST is the identity map on V.

Let $\{u_1, \dots, u_n\}$ be a basis for V. Then $\{T(u_1), \dots, T(u_n)\}$ is a basis for range T ~~since~~ T is injective. Extend $\{T(u_1), \dots, T(u_n)\}$ to a basis of W by adding $\{w_1, \dots, w_k\}$. For $w \in W$ $w = b_1 T(u_1) + \cdots b_n T(u_n) + c_1 w_1 + \cdots c_k w_k$. Define $S(w) = b_1 u_1 + \cdots b_n u_n$. Then $v = a_1 u_1 + \cdots + a_n u_n$ gives $ST(v) = S(a_1 T(u_1) + \cdots + a_n T(u_n)) = a_1 u_1 + \cdots + a_n u_n = v$

If $ST = I_V$ then ST is injective, shows that null $ST = \{0\}$, so null $T \subset$ null $ST = \{0\}$ and T is injective

(/6) 6. Prove or disprove that, if V is a 7 dimensional vector space, then every independent list of vectors in V of 7 vectors is a basis of V.

Suppose $\{v_1, \dots, v_7\}$ is an independent list of vectors in V. If it does not span, there is a vector $v \notin$ span $\{v_1, \dots, v_7\}$. $\{v_1, \dots, v_7, v\}$ is independent as $a_1 v_1 + \cdots + a_7 v_7 + bv = 0$ implies, if $b \neq 0$, that $b \in$ span $\{v_1, \dots, v_7\}$ a contradiction and if $b = 0$, then $\{v_1, \dots, v_7\}$ is not independent. Thus there is no such v, so that $\{v_1, \dots, v_7\}$ spans because $\{v_1, \dots, v_7, v\}$ has eight vectors, more than dim $V = 7$ a contradiction