MATH 108A HW 9 SOLUTIONS

RAHUL SHAH

Problem 1. [§5.20]

Solution. Let $\dim V = n$. By Proposition 5.20, we find that T has a diagonal matrix with respect to some basis of V. Thus by Proposition 5.21, we pick $\{v_1, \dots v_n\}$, a basis for V consisting of eigenvectors of T. Since each v_i is an eigenvector of T, $T(v_i) = \lambda_i v_i$. Since v_i is also an eigenvector for S, λv_i is also an eigenvector for S. Thus $ST(v_i) = \gamma_i \lambda_i v_i$, where $S(v_i) = \gamma_i v_i$. Thus $TS(v_i) = T(\gamma_i v_i) = \lambda_i \gamma_i v_i = ST(v_i)$. Since ST and TS agree on a basis for V, $ST = TS$. □

Problem 2. [§5.21]

Solution. Since $\dim V = \dim \operatorname{null} P + \dim \operatorname{range} P$, and $\operatorname{null} P + \operatorname{range} P$ is a subspace of V, we only need to show that $\operatorname{null} P \cap \operatorname{range} P = \langle 0 \rangle$ to complete the proof.

Let $v \in \operatorname{null} P \cap \operatorname{range} P$. Thus $P(v) = 0$. Since $v \in \operatorname{range} P$, for some $w \in V$, $P(w) = v$. However, $0 = P(v) = P^2(w) = P(w) = v$ and thus $v = 0$, completing the proof. □

Problem 3. [§5.22]

Solution. Let $v \in V$, where $v \neq 0$. Since $V = U \oplus W$, $v = u + w$, and this decomposition is unique. Assume that v is an eigenvector of $P_{U,W}$. Thus $P_{U,W}(v) = \lambda v$ for some $\lambda \in \mathbb{F}$. But $P_{U,W}(v) = u$ and thus $u = \lambda v = \lambda(u + w) = \lambda u + \lambda w$. Thus $(\lambda - 1)u + w = 0$. Since $V = U \oplus W$, $w = 0$ and $\lambda = 1$, or $u = 0$. We thus find two eigenvalues, 0 and 1, which have as eigenspaces U and W respectively. □

Problem 4. [§5.23]

Solution. Let $T \in \mathcal{L}(\mathbb{R}^4)$ be the transformation that has matrix:

$$\mathcal{M}(T) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Assume that T has an eigenvalue of $\lambda \in \mathbb{R}$. Notice that

$$\mathcal{M}(T - \lambda I) = \begin{pmatrix} -\lambda & 1 & 0 & 0 \\ -1 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & -1 & -\lambda \end{pmatrix},$$

and thus for $(x_1, x_2, x_3, x_4)^T \in \text{null}\,(T - \lambda I)$, we find the following equalities:

$$\begin{aligned} -\lambda x_1 + x_2 &= 0 \\ -x_1 - \lambda x_2 &= 0 \\ -\lambda x_3 + x_4 &= 0 \\ -x_3 - \lambda x_4 &= 0 \end{aligned}$$

We thus find that $x_1 = 0 = x_2$ and $x_3 = 0 = x_4$. Thus there is no non-zero vector v such that $T(v) = \lambda v$ and thus T has no eigenvalues. □

Problem 5. [§5.24]

Solution. Let V be a real vector space. Let $T \in \mathcal{L}(V)$ such that T has no eigenvalues and let U be a subspace of V such that U is invariant under T. Notice that $T \mid_U \in \mathcal{L}(U)$. If $\dim U$ is odd, then $T \mid_U$ must have an eigenvalue by Theorem 5.26. Thus for some non-zero $u \in U$, $T \mid_U (u) = \lambda u$ for some $\lambda \in \mathbb{R}$. However, $T(u) = T \mid_U (u) = \lambda u$ and thus T has an eigenvalue, which is a contradiction. Thus no subspace of V that is invariant under T can have be of odd dimension, and thus must be of even dimension. □