Problem 15 page 95

Prove that a is an eigenvalue of $p(T)$ if and only if $a = p(\lambda)$ for some eigenvalue λ of T

$\Rightarrow$ suppose a is an eigenvalue of $P(T)$ when

$p(z) = a_0 + a_1 z + \cdots + a_k z^k$

$p(T) = a_0 I + a_1 T + \cdots + a_k T^k$

$v \neq 0$, $a v = p(T) v \Rightarrow (P(T) - aI) v = 0 \Rightarrow$ not all injective $\Rightarrow$

$a_k (T - \lambda_1 I) \cdots (T - \lambda_k I) v = 0 \Rightarrow$ there is a λ_j eigenvalue of T root of $p(z) - a$

$\Rightarrow p(\lambda_j) - a = 0 \Rightarrow p(\lambda_j) = a$

$\Leftarrow$ if $a = p(\lambda)$ for some eigenvalue of T with eigenvector v

$P(T) v = (a_0 I + \cdots + a_k T^k) v = a_0 v + \cdots + a_k \lambda^k v$

$= (a_0 + a_1 \lambda + \cdots + a_k \lambda^k) v$

$= p(\lambda) v$

$\Rightarrow a = p(\lambda)$ is an eigenvalue of $P(T)$.