MATH 108A HW 7 SOLUTIONS

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Problem 1. [§5.1]

Solution. Let $U = U_1 + \ldots + U_m$, where U_i, a subspace of V, is an invariant subspace of $T \in \mathcal{L}(V)$. Let $u \in U$. By definition of U, $u = u_1 + \ldots + u_m$, where $u_i \in U_i$. Thus $T(u) = T(u_1) + \ldots + T(u_m)$. Since each U_i is invariant under T, $T(u_i) \in U_i$. Thus $T(u) \in U_1 + \ldots + U_m = U$, and thus U is invariant under T. □

Problem 2. [§5.2]

Solution. Let $\{U_\alpha\}_{\alpha \in J}$ for some set J be a collection of subspaces of V such that for a given $T \in \mathcal{L}(V)$, each U_α is invariant under T. Define

$$U = \bigcap_{\alpha \in J} U_\alpha.$$

To show that U is an invariant subspace of V, we must both show it to be a subspace of V and invariant under T. Since each U_α is a subspace, $\mathbf{0} \in U_\alpha$ for each $\alpha \in J$. Thus $\mathbf{0} \in \bigcap_{\alpha \in J} U_\alpha = U$. If we choose $u, v \in U$, then $u, v \in U_\alpha$ for each $\alpha \in J$. Sicne each U_α is a subspace of V, we see that $u + v \in U_\alpha$ for each U_α and thus $u + v \in U$. Similarly, U is closed under scalar multiplication. Thus we have shown that the intersection of any non-empty collection of subspaces of a vector space V is itself a subspace of V.

Let $u \in U$. Thus $u \in U_\alpha$ for each $\alpha \in J$. Since each U_α is invariant under T, $T(u) \in U_\alpha$ for each $\alpha \in J$ and thus $T(u) \in U$. Ergo, U is invariant under T. □

Problem 3. [§5.4]

Solution. Notice that since $T - \lambda I \in \mathcal{L}(V)$, null $(T - \lambda I)$ is a subspace of V. Let $v \in$ null $(T - \lambda I)$. Then $(T - \lambda I)(v) = \mathbf{0}$. Thus $T(v) = \lambda I(v) \Rightarrow T(v) = \lambda v$. Thus $ST(v) = S(\lambda v) = \lambda(v)$. Since $ST(v) = TS(v)$, we find that $T(S(v)) = \lambda(S(v))$. This $T(S(v)) - \lambda S(v) = 0$ and thus

$$(T - \lambda I)(S(v)) = \mathbf{0}.$$

Thus $S(v) \in$ null $(T - \lambda I)$, thus proving that null $(T - \lambda I)$ is invariant under S. □

Problem 4. [§5.5]

Solution.

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For a given $\lambda \in \mathbb{F}$, to find the eigenvectors associated to λ, we must find null $(T - \lambda I)$. But

$$T - \lambda I = \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}.$$

Thus

$$(T - \lambda I)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Setting this equal to $\mathbf{0}$, we find:

$$-\lambda x_1 + x_2 = 0 \tag{4.1}$$

$$x_1 - \lambda x_2 = 0 \tag{4.2}$$

Solving for λ, we find that $\lambda = \pm 1$. For $\lambda = 1$, we find that

$$\text{null } (T - I) = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle,$$

and for $\lambda = -1$, we find

$$\text{null } (T + I) = \left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle.$$

□

Problem 5. [§5.7]

Solution. Just as in the last problem, we find, for a fixed $\lambda \in \mathbb{F}$, null $(T - \lambda I)$. Notice that we find the following system of equations:

$$\begin{aligned} (1-\lambda)x_1 + x_2 + \ldots + x_n &= 0 \\ x_1 + (1-\lambda)x_2 + \ldots + \ldots x_n &= 0 \\ &\vdots \\ x_1 + x_2 + \ldots + (1-\lambda)x_n &= 0 \end{aligned}$$

We thus find that $x_1 + x_2 + \ldots + x_n = \lambda x_1 = \lambda x_2 = \ldots = \lambda x_n$. Thus $x_1 = x_2 = \ldots = x_n$. We thus find that $n \cdot x_i = \lambda x_i$ and thus n is the only eigenvalue for this operator. We also find that

$$\text{null } (T - nI) = \left\langle (1, 1, \ldots 1)^T \right\rangle.$$

□

Problem 6. [§5.8]

Solution. Since $T(z_1, z_2, \ldots) = (z_2, z_3, \ldots)$, if $(x_1, x_2, \ldots)$ is an eigenvector of T, then for some $\lambda \in \mathbb{F}$, $x_{i+1} = \lambda x_i$. Thus for any $\lambda \in \mathbb{F}$, λ is an eigenvalue of T and the associated eigenvectors are the vectors in

$$\left\langle (1, \lambda, \lambda^2, \ldots) \right\rangle.$$

□

Problem 7. [§5.9]

Solution. If T is not injective, then $\lambda_0 = 0$ is an eigenvalue. If there are at least $k+1$ other distinct eigenvalues, $\lambda_1, \dots \lambda_{k+1}$, then we can find eigenvectors $v_1, \dots v_{k+1}$ such that $T(v_i) = \lambda v_i$ for $i \leq k+1$. By Theorem 5.6, $v_1, \dots v_{k+1}$ are linearly independent. However, $A = \{\lambda_1 v_1, \dots \lambda_{k+1} v_{k+1}\} \subset \text{range}\,(T)$. Since $\{v_1, \dots v_{k+1}\}$ is linearly indepdendent, so is A. However, no set of $k+1$ vectors in range (T) can be linearly independent, since $\dim \text{range}\,(T) = k$. Thus we can have at most $k+1$ distinct eigenvalues for T. □

Problem 8. [§5.10]

Solution. Let $\lambda \in \mathbb{F} \setminus \{0\}$ and $T \in \mathcal{L}(V)$, where T is invertible.

- '$\Rightarrow$' Assume λ is an eigenvalue of T. Then for some $v \in V$ $(v \neq 0)$, $T(v) = \lambda v$. Thus $T^{-1}(\lambda v) = v = \frac{1}{\lambda} \cdot (\lambda v)$. Thus $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.
- '$\Leftarrow$' If $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}, there is some $v \in V$ $(v \neq 0)$ such that $T^{-1}(v) = \frac{1}{\lambda} v$. Thus $T(\frac{1}{\lambda} v) = v = \lambda \cdot (\frac{1}{\lambda} v)$. Thus λ is an eigenvalue for T.

□

Problem 9. [§5.11]

Solution. Assume $\lambda \in \mathbb{F}$ is an eigenvector for $S \circ T$. Then for some $v \in V$ $(v \neq 0)$, $S \circ T(v) = \lambda v$. Then $T \circ S(T(v)) = T(\lambda v) = \lambda T(v)$. Thus λ is an eigenvalue of $S \circ T$. By symmetry, $S \circ T$ and $T \circ S$ have the same eigenvalues. □

Problem 10. [§5.14]

Solution. Let $\mathfrak{p} \in \mathcal{P}(\mathbb{F})$ be a polynomial where $\mathfrak{p}(T) = a_0 I + a_1 T + \ldots + a_m T^m$. Then $\mathfrak{p}(STS^{-1}) = a_0 I + a_1(STS^{-1}) + \ldots + a_m(STS^{-1})^m = a_0 SIS^{-1} + a_1(STS^{-1}) + \ldots + a_m(ST^m S^{-1}) = S\mathfrak{p}(T)S^{-1}$, by linearity of $\mathfrak{p}$. □