Mathematics 108A HW 1 (selected solutions)

1. Show that $\mathbb{Z}_2 = \{0,1\}$ is a field

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

shows that $\mathbb{Z}_2$ satisfies all required properties. In particular $-1 = 1$ as $1+1=0$

$\mathbb{Z}_3$ is also a field

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Shows $-1 = 2$ and $-2 = 1$ as $1+2=0$

shows $2^{-1} = 2$ as $2 \cdot 2 = 1 \bmod 3$

$\mathbb{Z}_4$ is not a field as 2 does <u>not</u> have a multiplicative inverse: $2\cdot 1 = 2$, $2 \cdot 2 = 0$, $2 \cdot 3 = 2$

4. $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \; x \geq 0, y \geq 0 \right\}$

(i) if $u = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $v = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in V$ then $u+v = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} \in V$

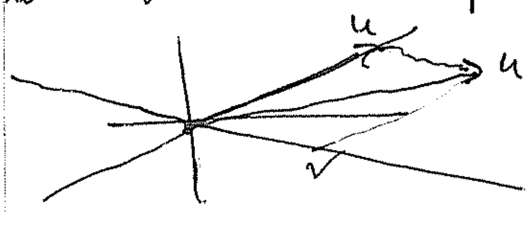
as $x_1 + x_2 \geq 0$ if x_1 and $x_2 \geq 0$

$y_1 + y_2 \geq 0$ if y_1 and $y_2 \geq 0$

(ii) not true as $(-1)\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -y_1 \end{pmatrix}$ and $-x_1 \leq 0$ and $-y_1 \leq 0$.

(iii) V is <u>not</u> a vector space as it is not closed under scalar multiplication.

7. two lines through the origin in $\mathbb{R}^2$ are closed under scalar multiplication but <u>not</u> addition.



$u+v$ is not in the two lines.

10. If U_1, U_2 and W are subspace of a vector space V such that $U_1 + W = U_2 + W$ then $U_1 = U_2$.

Not true! Suppose $U_1 = \{(r,0) \mid r \in \mathbb{R}\}$, $U_2 = \{(0,s) \mid s \in \mathbb{R}\}$ and $W = \{(v,u) \mid u, v \in \mathbb{R}) = \mathbb{R}^2$

Then $U_1 + W = \mathbb{R}^2 = U_2 + W$ but $U_1 \neq U_2$.

11. If $U_1 \oplus W = U_2 \oplus W$, then $U_1 = U_2$

Let $U_1 = \{(0,s) \mid s \in \mathbb{R}\}$, $U_2 = \{(s,s) \mid s \in \mathbb{R}$ and $W = \{(r,0) \mid r \in \mathbb{R}\}$ then

$$U_1 \oplus W = \mathbb{R}^2 = U_2 \oplus W \text{ but } U_1 \neq U_2$$

Note: These are very sketchy hints at solutions. For full credit you would want to include more detail and an expanded discussion