1. (4 points) Consider the following vector space:

\[ V = \{ x \in \mathbb{R}^3 \mid x_1 - 2x_2 + 3x_3 = 0 \} \]

Find a basis for \( V \) and determine its dimension.

\( V \) is the set of \( x = (x_1, x_2, x_3) \) that are solutions to the given system of linear equations, so let's find the solutions.

\[
\begin{pmatrix}
1 & -2 & 3 & | & 0 \\
2 & 1 & -1 & | & 0 \\
\end{pmatrix} \Rightarrow \begin{pmatrix}
1 & -2 & 3 & | & 0 \\
0 & -5 & 7 & | & 0 \\
\end{pmatrix}
\]

\[ \begin{align*}
x_1 - 2x_2 + 3x_3 &= 0 \\
-5x_2 + 7x_3 &= 0 \\
\end{align*} \]

\[ \Rightarrow \begin{cases} x_1 = 15x_3 \\
x_2 = 7/5x_3 \\
x_3 = x_3 \\
\end{cases} \]

\[ V = \text{solns} = \{ r \begin{pmatrix} 0 \\ 7/5 \\ 1 \end{pmatrix} : r \in \mathbb{R} \} = \{ s \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} : s \in \mathbb{R} \} \]

Claim: \( \mathcal{B} = \{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \} \) is a basis of \( V \).

For \( \mathcal{B} \) to be a basis, the set of vectors in \( \mathcal{B} \) must be (1) linearly independent and (2) a spanning set of \( V \).
1) Check linear independence.
Suppose \[ a\left(\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right) = \left(\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right) \]
a = 0 is the unique solution for this eqn.
(i.e. ——— only ——— ——— ———).

So the set \( \left\{ \left(\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right) \right\} \) is linearly independent.

NB Any time you have a set w/ 1 vector, if that vector is not zero, the set is linearly independent.

2) Check that \( B \) spans \( V \).
\[ \text{span } B = \text{span } \left\{ \left(\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right) \right\} \]
= all linear combinations of \( \left(\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right) \)
= \[ \left\{ r \left(\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right) : r \in \mathbb{R} \right\} \]
= solns = \( V \) by our previous calculation