Theorem 1. (Green’s Theorem) Let $D$ be a region in $\mathbb{R}^2$ that satisfies Assumption 8.1, and let $\partial D$ be its positively oriented boundary. If $\vec{F}(x,y) = (P(x,y), Q(x,y))$ is a $C^1$ vector field on $D$, then

$$\int_{\partial D} \vec{F} \cdot d\vec{s} = \int \int_D \left( \frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y} \right) dA$$

Theorem 2. (Divergence Theorem) Let $W$ be a region in $\mathbb{R}^3$ that satisfies Assumption 8.2, and let $\partial W$ be its positively oriented boundary. Then

$$\int \int_{S = \partial W} F \cdot dS = \int \int \int_W \text{div} F dV$$

for a $C^1$ vector field $\vec{F}$ defined on $W$.

Theorem 3. (Stoke’s Theorem) Let $S$ be a parameterized surface by a one-to-one parameterization $\vec{r} : D \subset \mathbb{R}^2 \to \mathbb{R}^3$, where $D$ is a region to which Green’s Theorem applies (i.e. satisfies Assumption 8.1). Let $\partial S$ be the positively oriented piecewise smooth boundary of $S$. Then

$$\int_{\partial S} \vec{F} \cdot d\vec{s} = \int \int_S \text{curl} \vec{F} \cdot d\vec{S}$$

for a $C^1$ vector field $\vec{F}$ defined on $S$.

### Integrals

<table>
<thead>
<tr>
<th>Integrate</th>
<th>$\int_{\text{over a curve, } \vec{c}(t)} f , ds$</th>
<th>$\int_{\text{over a surface, } S \text{ parameterized by } \vec{r}(u,v)} f , dS$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a scalar valued function, $f$</td>
<td>$\int_a^b f(\vec{c}(t))</td>
<td></td>
</tr>
<tr>
<td>a vector-valued function, $\vec{F}$</td>
<td>$\int_{a}^{b} F(\vec{c}(t)) \cdot \vec{c}'(t) dt$</td>
<td></td>
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</tbody>
</table>
Practice Problems:
Set up each integral both using a classical integration theorem from Chapter 8 and without a classical integration theorem. Cite the integration theorem that applies by name, and integrate using whichever method you prefer. Use clear presentation and justify every step as if this were an exam.

1. \( \int_{\bar{c}} \bar{F} \cdot d\bar{s} \) where \( \bar{F} = e^x y \hat{j} - e^x y \hat{i} \) and \( \bar{c} \) is the curve that forms the boundary of the triangle defined by the lines \( y = 0 \), \( x = 1 \), and \( y = x \) oriented counterclockwise

2. \( \int_S = \partial W \int S \bar{F} \cdot d\bar{S} \) where \( \bar{F}(x,y,z) = (-e^x \cos(y)) \hat{i} + (e^x \sin(y)) \hat{j} + \hat{k} \) and \( S \) is the surface of the sphere \( x^2 + y^2 + z^2 = 1 \)

3. \( \int_{\partial S} \bar{F} \cdot d\bar{s} \) where \( \bar{F}(x,y,z) = -2y \hat{i} + z \hat{j} - z \hat{k} \) and \( \bar{c} \) is the curve defined by the intersection of the cylinder \( z^2 + x^2 = 1 \) and the plane \( y = x + 1 \), oriented counterclockwise as seen from the origin

4. \( \int_{\partial S} \bar{F} \cdot d\bar{s} \) where \( \bar{F}(x,y,z) = (2x + y) \hat{i} + (2y - x) \hat{j} \) and \( \bar{c} \) is the helix \( \bar{c}(t) = (\cos(t), \sin(t), t), t \in [0, 3\pi] \), followed by the line segment from \((-1, 0, 0)\) back to \((1, 0, 0)\)

5. \( \int_S = \partial W \int S \bar{F} \cdot d\bar{S} \) where \( \bar{F}(x,y,z) = (x + y^2 + 1) \hat{i} + (y + xz) \hat{j} \) and \( S \) consists of the part of the cone \( z^2 = x^2 + y^2 \) bounded by the disks \( 0 \leq x^2 + y^2 \leq 1 \), \( z = 1 \), and \( 0 \leq x^2 + y^2 \leq 4 \), \( z = 2 \)

6. \( \int_{\partial S} \bar{F} \cdot d\bar{s} \) where \( \bar{F} = (2x + 3y + 2) \hat{i} - (x - 4y + 3) \hat{j} \) and \( \bar{c} \) is the ellipse \( x^2 + 4y^2 = 4 \) oriented clockwise

7. Which problems from last week can we do using a classical integration theorem? Which theorem applies? Why does the theorem apply or not apply in each instance? If you can apply a classical integration theorem, rework the problem using our new methods.

8. Go back to each of the previous problems, and solve using the other method to check your answer.