Linear Algebra Worksheet 8
Math 108A Fall 2009, TA Grace Kennedy

NAME:________________________________
Course Website: http://math.ucsb.edu/∼kgracekennedy/F09108A.html
Supplemental Reading: Axler Pages 1-93

FINAL EXAM REVIEW

This is the coversheet for the final exam:

Instructions:

- This exam consists of 8 problems for a total of 72 points on 9 pages.
- You MUST write your name legidibly at top of each page of the exam, in the space provided. You may use the space on the back of a page but ONLY for the problems on the front of that page. You may want to work our your solutions first on scratch paper so that you can write them neatly on the test. If so, scratch paper is available from the proctors.
- No books, notes, or calculators maybe used on this exam.
- You must show all your work and fully justify your answers in order to receive full credit.
- If you are asked to prove a result from the book, including homework, or from the lectures, you must give a full proof of that result. You may use another method than that used in the book or in lecture, if your wish.
- If your answer uses a result proved in the book or in class, you must summarize what the result says and, if necessary, explain how you are using it.
- If there are multiple sections to a question, you may use results of those earlier questions that are true, even if you cant prove them, to do later parts of the question.
- Your answers will be graded for clarity and organization as well as mathematical correctness.
- Your final course grade will be no lower than the grade on this final exam. A composite course grade (using the midterms, homework, etc) will be computed and you receive the better of these two grades.
- If you have any questions during the final exam, please address them to the proctors.

Let $\mathbb{F}$ be a field, and let V be a vector space of dimension $n < \infty$ over $\mathbb{F}$. Unless otherwise stated, all vector spaces are over this field $\mathbb{F}$.

1 Fields and Vector Spaces

1. *Cardinality vs. Dimension of a Vector Space.* Prove or give a counterexample for the following statements:
   (a) A finite dimensional vector space has finitely many vectors.
   (b) An infinite dimensional vector space has infinitely many vectors.
   (c) All vector spaces have infinitely many elements.
   (d) All vector spaces over finite fields have finitely many elements.
   (e) Assume V and W are vector spaces such that $|V| = |W|$, i.e. V and W have the same cardinality. V and W are isomorphic.
   (f) If V and W are vector spaces such that $dimV = dimW$, then V and W are isomorphic
   (g) Assume that V and W are vector spaces where neither need be finite dimensional. For any arbitrary vector space V, B is a basis for V if for each $v \in V$, $v = a_1v_1 + a_2v_2 + \ldots a_kv_k$ for $v_i \in B$, and some $k \in \mathbb{N}$, and if any finite subset of B is linearly independent. If V and W have bases B_1 and B_2, where $|B_1| = |B_2|$. Then V and W are isomorphic.

2. Let $\mathbb{Q}$ be the rational numbers and $\mathbb{Q}[\sqrt{2}] = \{p + q\sqrt{2} | p, q \in \mathbb{Q}\}$.
   (a) Prove or disprove: $\mathbb{Q}[\sqrt{2}]$ a vector space over $\mathbb{Q}$?
   (b) Is $\mathbb{Q}[\sqrt{2}]$ a vector space over $\mathbb{R}$?
   (Convince yourself of your answer, perhaps using your write up to part a) to the point you could prove this on an exam.)
   (c) Prove or disprove: $\mathbb{Q}[\sqrt{2}]$ is a vector space over $\mathbb{Q}[\sqrt{3}]$.

3. Prove: $(-1)v = -v$ and $-(-1) = 1$.

2 Linear Independence, Span, Basis

1. Prove or disprove: Any subset of a set of linearly independent vectors is itself linearly independent.

2. Prove or disprove: A subset of a spanning set of vectors of a vector space can also span the vector space. Is this true or false if we say "any subset of a spanning set spans the vector space?"

3. Prove: $\{v_1, \ldots, v_n\}$ is a basis of $V \Leftrightarrow \{v_1, \ldots, v_n\}$ is linearly independent $\Leftrightarrow \{v_1, \ldots, v_n\}$ spans V.

4. Let U be the subspace of $\mathbb{R}^4$ defined by

$$U = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 = x_2 \text{ and } 3x_3 = 7x_4\}$$

   Find a basis of U. What is its dimension?. Complete the basis you found for U to form a basis of $\mathbb{R}^4$.

5. Let U be a subset of V, a vector space. Show that $span(U)$ is a subspace of V. Assume that $|V| \geq |B|$, where B is a basis for V. Is $span(U) = V$? Prove or give a counterexample.

6. Let V be the collection of all poynomials with coefficients in $\mathbb{R}$. Show that V is a vector space over $\mathbb{R}$.

7. Let B be the collection of polynomials given as follows: $\{1, x-1, x^2 - x, x^3 - x^2, \ldots\}$. Notice that $1 + (x-1) + (x^2 - x) + \ldots = 0$. Show that B is a basis for V. Explain why this isn't a contradiction.

8. Let $V = \mathbb{R}^5$. Let $U = span((1,0,1,0,1), (3,0,1,3,1), (2,1,4,3,5), (1,1,1,1,1), (1,0,3,2,4))$.
   (a) Find $\dim U$
   (b) Find an operator $T \in L(V)$ such that $range(T) = U$.
   (c) Find a basis B of V and an operator $T \in L(V)$ such that for each $v \in B$, $T(v) = v$ or $T(v) = 0$.

3 Linear Transformations

1. Let V and W both be vector spaces of dimension m and n respectively over $\mathbb{F}$, and let $T : V \to W$ be a linear transformation. Prove there are bases $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$ for V and W respectively such that there is a value $k \leq n$ with

$$T(v_i) = \begin{cases} w_i, & \text{for } i \leq k \\ 0, & \text{for } i > k. \end{cases}$$

2. Show that the linear combination of linear transformations is a linear transformation.
   (How would this fit into a proof that the set of linear transformations of a vector space for a vector space themselves?)

3. Consider the following function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by:

$$\begin{array}{ccc} e_1 & \mapsto & e_2 \\ e_2 & \mapsto & e_1 \end{array}$$

   (a) What is Tv for an arbitrary element of $\mathbb{R}^2$? Is a linear transformation? Why or why not?

(b) Write T as a matrix.

(c) Find a basis for the null space. (You might need to consult the technicality discussed after Corollary 2.11 on page 29.)

(d) Find a basis for the range.

(e) Is this linear transformation an isomorphism? If so, give its inverse. If not, show why.

(f) Find all eigenvalues and eigenvectors.

4. Assume that V is a finite dimensional vector space over $\mathbb{F}$.

   (a) Find a vector space V such that $L(V)$ is isormorphic to V.

   (b) Assume V is a vector space such that $L(V)$ is not isomorphic to V. Is it possible that $L(L(V))$ is isomorphic to V?

4 Polynomials

1. After applying the division algorithm to $s(x)$ and $p(x)$ and getting $p(x) = s(x)q(x) + r(x)$, is it true that $\deg p(x) = \deg s(x) \deg q(x) + \deg r(x)$? If true, explain why, if not, correct and prove the correct statement.

2. Suppose $T : \mathbb{R}^7 \to \mathbb{R}^7$. What are the possible numbers of real eigenvalues?

3. Prove or disprove: The set of polynomials having only terms of odd degree along with zero form a vector space.

   (a) Give a basis. What is the dimension?

   (b) Give a basis so that each basis vector has at least two distinct powers of x.

   (c) Does there exist a basis so that each basis vector does not have a zero at $x = 0$?

4. Let V be the vector space of polynomials of degree less than or equal to 2.

   (a) What is the dimension?

   (b) Give a basis. (Do not forget to prove linear independence and spanning.)

   (c) Does there exist a basis so that each basis vector evaluated at $x = 1$ is zero?

   (d) Give a basis so that each basis vector has at least two distinct powers of x.

   (e) Find a linear transformation between these two bases.

5. Fix $n \in \mathbb{N}$.

(a) Show that the collection of polynomials that have degree less than or equal to n, V, is a vector space over $\mathbb{F}$.

(b) Let $T : V \to V$ be defined as $T(1) = x$, $T(x) = x^2$, $\ldots T(x^n) = 1$. Is T a linear transformation on V?

(c) How about is T is the same as above except $T(x^{n-1}) = 1$. How about if if $T(x^2) = x^4$?

(d) Notice that F is a 1-dimensional vector space over itself. Show that $L(F)$ is a subspace of V.

This last problem wasn't meant to be too hard, but to give a nice justification to the term 'linear' algebra.

5 Invariant Subspaces, Eigenvalues and Eigenvectors

1. Prove or disprove: There exists a $T \in \mathcal{L}(\mathbb{C}^2)$ so that T has no eigenvalues, and hence no eigenvectors.

2. Can you find a vector space W, and a linear transformation T on T, such that T has no eigenvalues but T^2 has eigenvalues? Can you find a T that has eigenvalues where T^2 has none? Prove your answers.

3. Is Theorem 5.24 true if we were in an infinite dimensional vector space? Explain why you think so or give a counter example.

4. Return to your homework proof that an eigenvalue of ST is an eigenvalue of TS. Does this work if the eigenvalue is 0?

5. Prove or Disprove: If S and T are invertible, then an eigenvector of ST is an eigenvector of TS.

6. We will consider an element of $\mathcal{L}(Mat_{3\times3}(\mathbb{R}))$. Let $A \in Mat_{3\times3}(\mathbb{R})$ be the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   (a) Note, $\mathcal{L}(Mat_{3\times3}(\mathbb{R}))$ is a vector space! What is the dimension of $\mathcal{L}(Mat_{3\times3}(\mathbb{R}))$?

   (b) Show that $T : Mat_{3\times3}(\mathbb{R}) \to Mat_{3\times3}(\mathbb{R}) \in \mathcal{L}(Mat_{3\times3}(\mathbb{R}))$ given by $T(X) = \frac{1}{2}(AX + XA)$ is a linear transformation. (First convince yourself that it does take "vectors" in $Mat_{3\times3}(\mathbb{R})$ to vectors in $Mat_{3\times3}(\mathbb{R})$.)

(c) Calculate T of the following matrix

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

(d) Let M_{ij} be the 3×3 matrix with 1 in the ijth place and 0's everywhere else. Show M_{ij} is an eigenvector of T. What is the associated eigenvalue?

7. Give a complete proof of Theorem 5.6.