The Role of Continued Fractions in Rediscovering a Xenharmonic Tuning

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10/11/2012

Outline

Motivation

Wendy Carlos

- American composer and Grammy winner

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- American composer and Grammy winner
- Helped to popularize and improve the Moog synthesizer
- Used creative and unconventional tunings in original compositions

Carlos’ Best Known Works: Soundtracks

1971

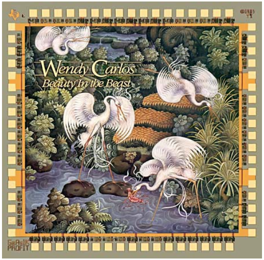
1980

1982

Carlos' Best Known Works: Albums

1968

1984



1986

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Physics

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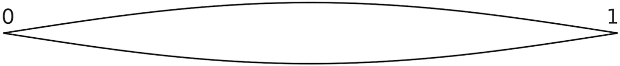
- The (fundamental) frequency $f = 1/T$ (in Hz $= 1/$sec) is the number of cycles per second.
- If L is the length of the string, then

$$f = \frac{v}{2L}$$

where v depends only on the density and tension.

Harmonics (Octaves Give Same “Note”)

Standing Wave	Frequency
	f

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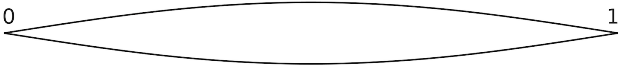
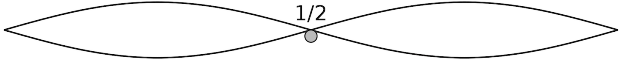
	f
	$2f \sim f$

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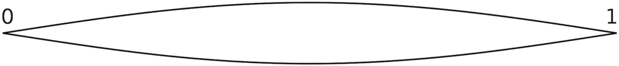
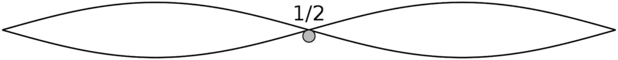
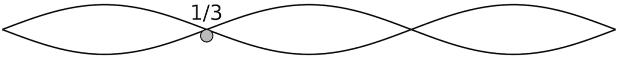
	f
	$2f \sim f$
	$3f$

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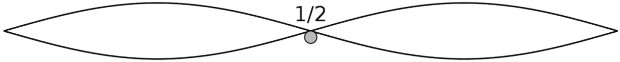
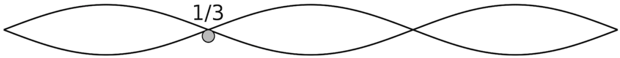
	$2f \sim f$
	$3f$
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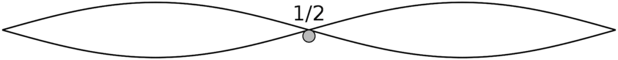
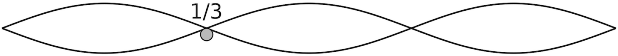
	$2f \sim f$
	$3f$
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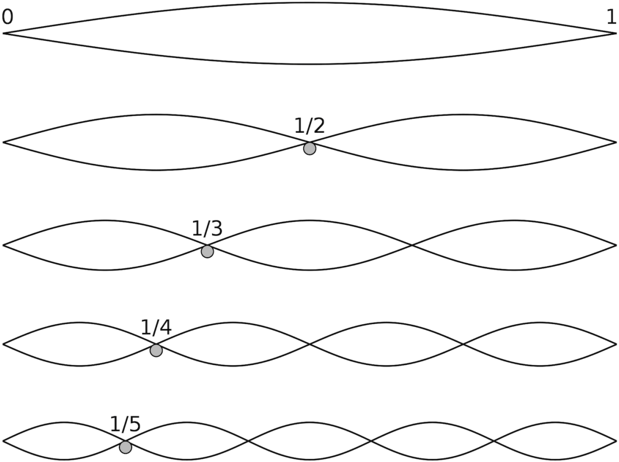	f $2f \sim f$ $3f$ $4f \sim 2f \sim f$ $5f$

Likewise, $6f \sim 3f$, $8f \sim f$, $10f \sim 5f$, etc.

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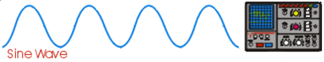	Link
	Link
	Link

Table: Different instruments playing the same frequency

A Little Functional Analysis

In the Hilbert space $L^2([0, 2\pi])$, $\exists$orthonormal decompositions

$$h(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where $|c_n| = |\langle h(x), e^{inx} \rangle| =$ length of the projection onto e^{inx}.

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If $h(x)$ is real and odd, then Euler's formula implies

$$h(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$$

where $a_n = 2ic_n \in \mathbb{R}$ is the amplitude of the nth harmonic.

Example: Sawtooth Wave

Consider the 2π-periodic function $s(x)$ s.t.

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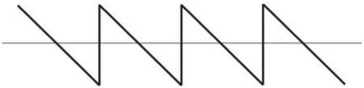

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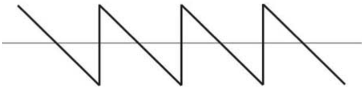

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Analog synthesizers use sawtooths via subtractive synthesis.

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If we plug in the sawtooth $s(x)$,

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{n=-\infty}^{\infty} |c_n|^2 = ||s(x)||^2$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 dx = \frac{\pi^2}{12}$$

Back to Frequencies and Notes

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There is an equivalence relation on frequencies $f, g \in (0, \infty)$:

$$f \sim g \Leftrightarrow f = 2^n g \text{ for some } n \in \mathbb{Z}$$

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In general,

$$[f] \mapsto \log_2(f) + \mathbb{Z} \mapsto e^{2\pi i \log_2(f)}$$

“Circle” of Fifths

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It's the most significant interval behind the octave.

The Perfect Fourth, Major Third, . . .

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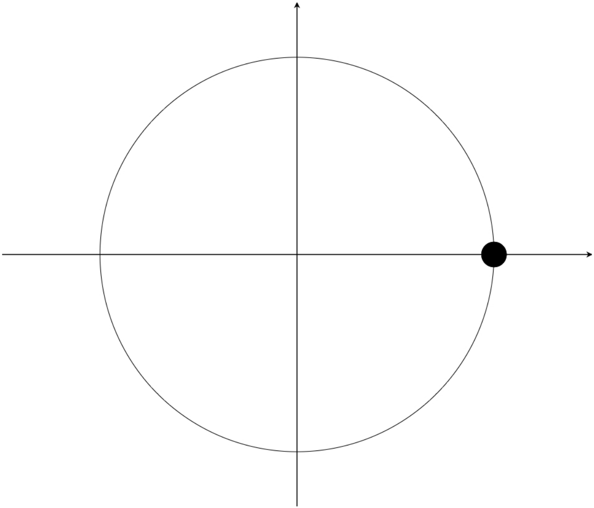
The fifth harmonic gives rise to the **major third** $5/4$.

Stacking Perfect Fifths



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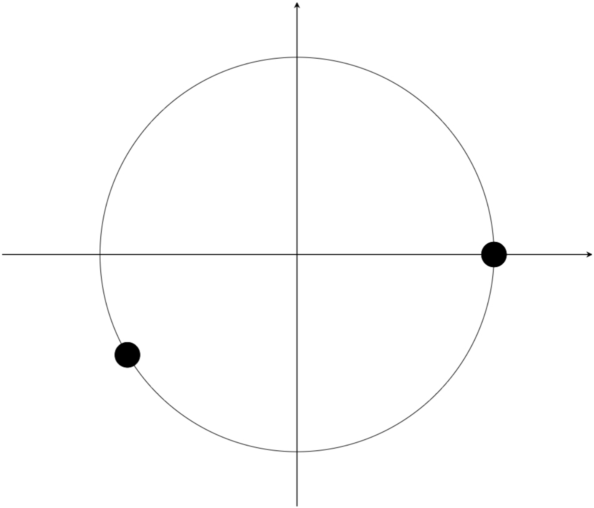

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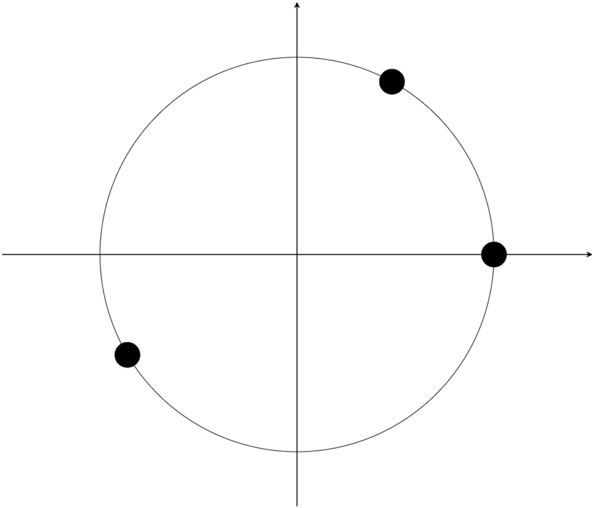

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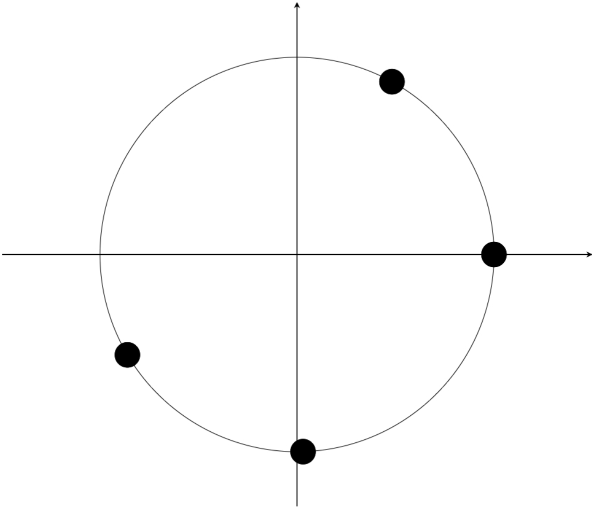

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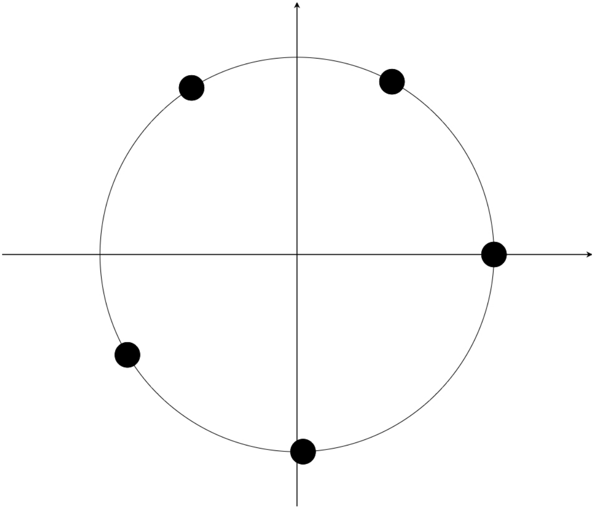

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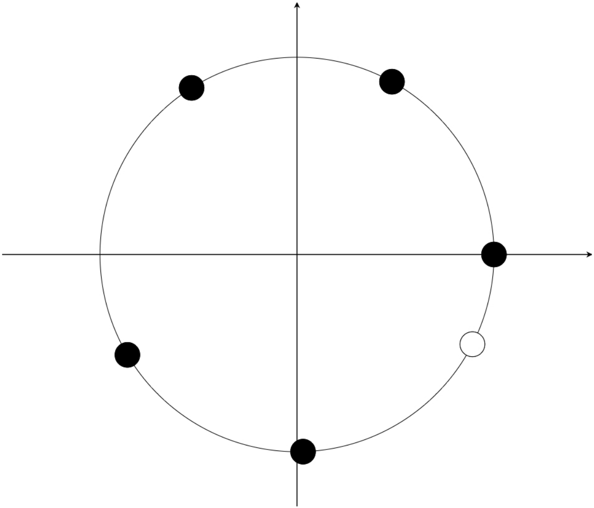

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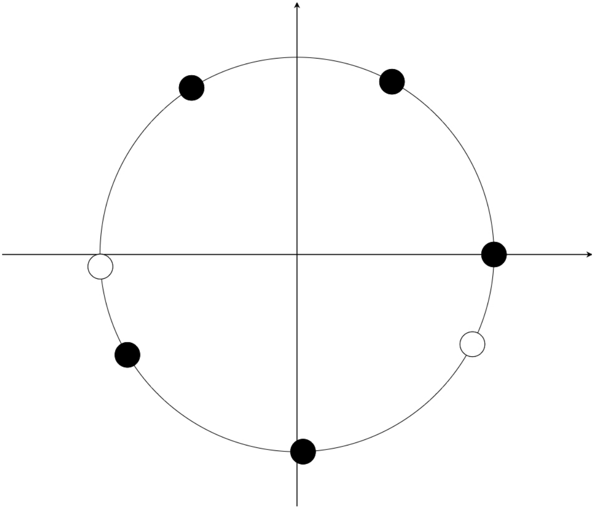

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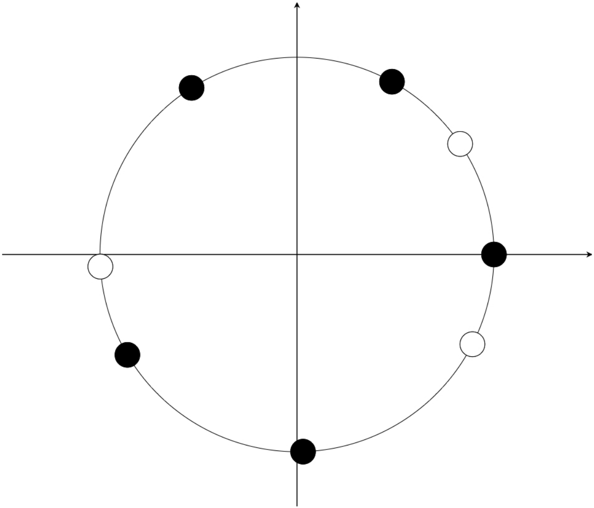

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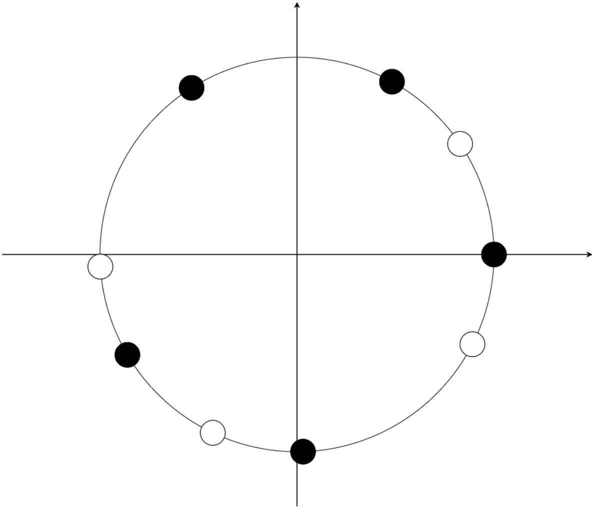

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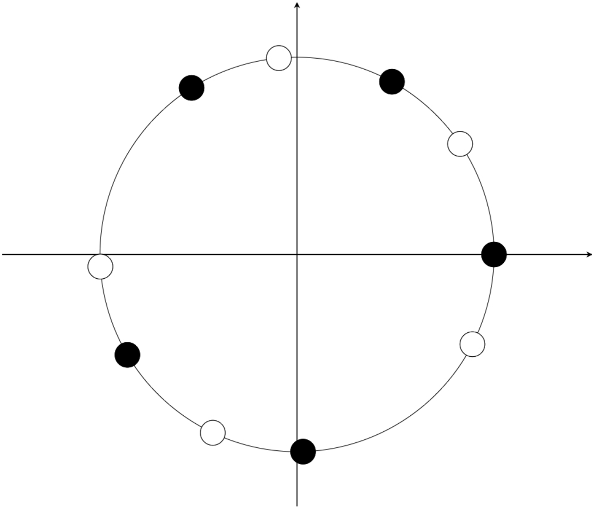

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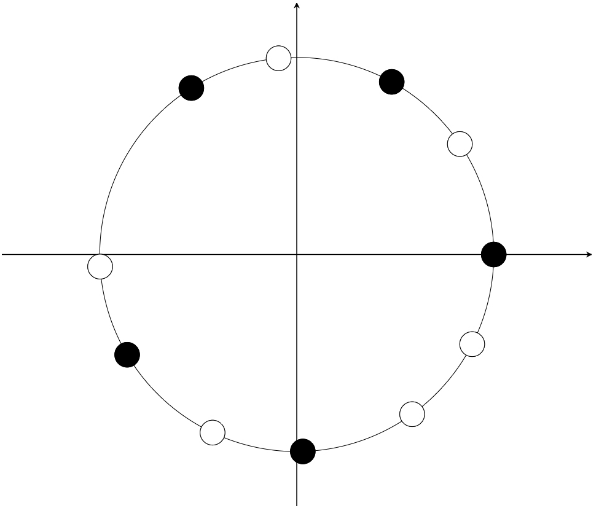

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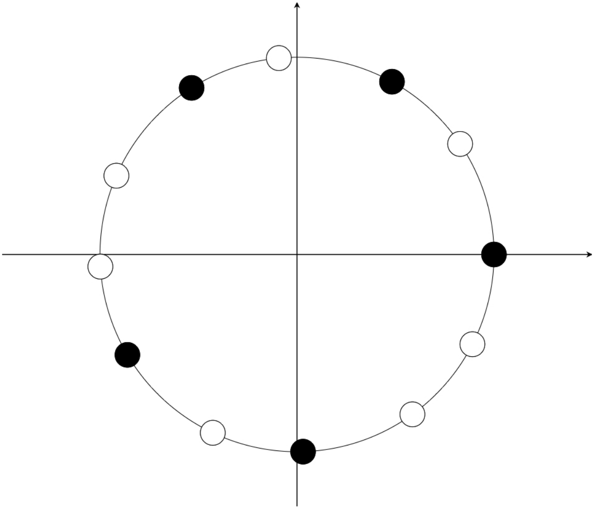

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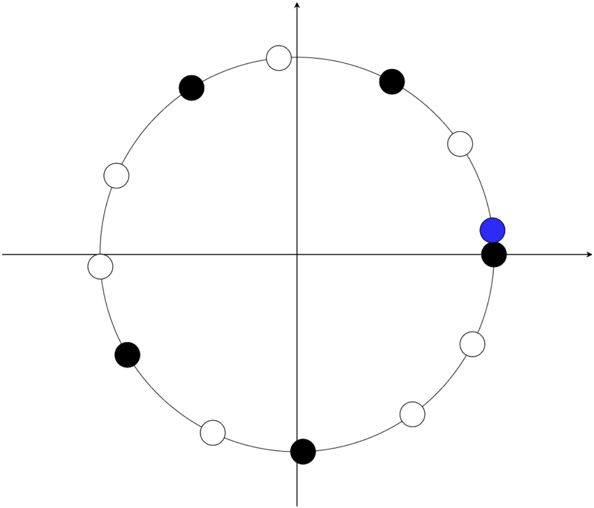

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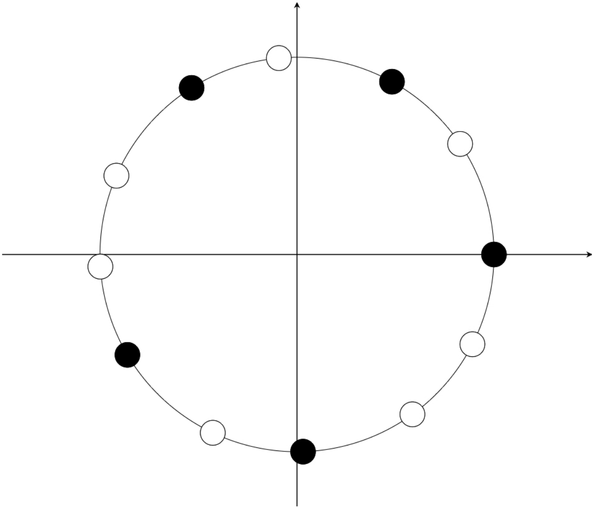
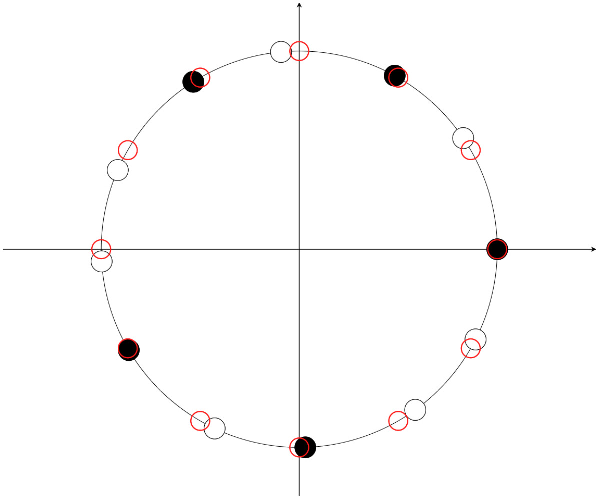
$\mathbb{Q}$ is dense $\mathbb{R}$: we can get arbitrarily good rational approximations a/b to $\log_2(3/2)$.

$e^{2\pi i/b}$ will generate a cyclic subgroup in $\mathbb{S}^1$ corresponding to a division of the interval $(1, 2]$ into b equal pieces.

Let's Stop at 12 . . . Why?

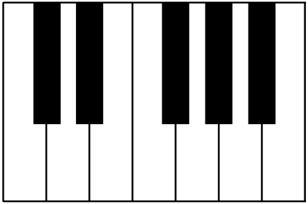


12-Tone Equal Temperment Scale



1200 cents = whole interval, so notes are 100 cents apart.

One Octave on a Standard Keyboard



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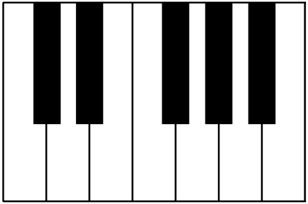

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Ideally, we'd want to divide the interval into as few pieces as possible while getting a good approximation to the perfect fifth.

Continued Fractions

Simple Continued Fractions

$$[a_0; a_1, a_2, \ldots, a_n] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots \cfrac{1}{a_n}}}$$

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Theorem

The infinite continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} := \lim_{n\to\infty} [a_0; a_1, a_2, \ldots, a_n]$$

converges if and only if the sum $\sum_{i=0}^{\infty} a_i$ *diverges.*

Unique Expansions

Theorem

Let $\alpha \in \mathbb{R}$. There is a unique† *continued fraction expansion*

$$\alpha = [a_0; a_1, a_2, \ldots]$$

s.t. $a_0 \in \mathbb{Z}$ and $a_1, a_2, \ldots$ are positive integers.

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The continued fraction expansion of $\alpha \in \mathbb{R}$ as above is infinite if and only if α is irrational.

The continued fraction expansion is eventually periodic if and only if α is a quadratic irrational.

Examples of Continued Fraction Expansions

$$-\frac{13}{5} = [-3; 2, 2]$$

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There is no known pattern in the expansion of π.

We don't even know whether or not the terms in the expansion of $\sqrt[3]{2}$ are bounded.

Convergents

Definition

The *n***th convergent** of $\alpha = [a_0; a_1, a_2, \ldots]$ is

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n]$$

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with p_n, q_n relatively prime integers ($q_n > 0$).

Theorem

Suppose $\alpha \notin \mathbb{Q}$. Then the convergents p_n/q_n are best approximations (and vice versa) in the following sense:

If $a/b \in \mathbb{Q}$ is written is lowest terms and $b < q_n$, then

$$|b\alpha - a| > |q_n\alpha - p_n|$$

Combinatorics of 12-Tone Chromatic Scale

$$\log_2(3/2) = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{3 + \cdots}}}}}$$

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$$\frac{1}{2} < \frac{7}{12} < \frac{31}{53} < \ldots < \log_2(3/2) < \ldots < \frac{24}{41} < \frac{3}{5} < 1$$

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$2^{7/12} \approx 3/2$ is good, but $2^{4/12} \overset{?}{\approx} 5/4$ is not as good.

A Tuning of Wendy Carlos

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We get a new equivalence relation on frequencies:

$$f \sim g \Leftrightarrow f = (3/2)^n g \qquad \text{for some } n \in \mathbb{Z}$$

How Carlos Chose Divisions

She picked some notes (including the major third) that she wanted to be well approximated.

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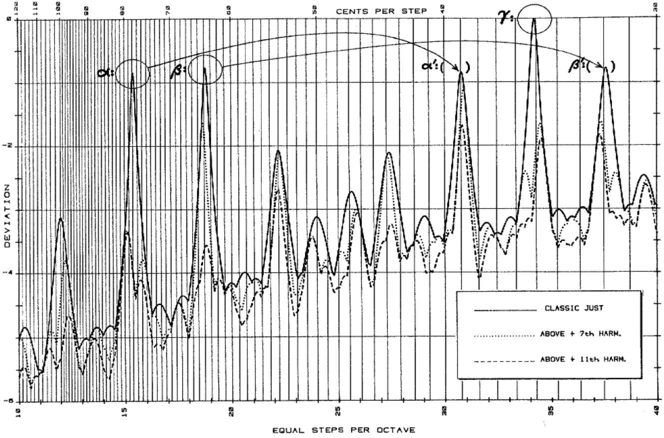

α, β, and γ Scales

She found the following desirable divisions

step sizes	number of pieces
$\alpha =$ 77.995... cents	9
$\beta =$ 63.814... cents	11
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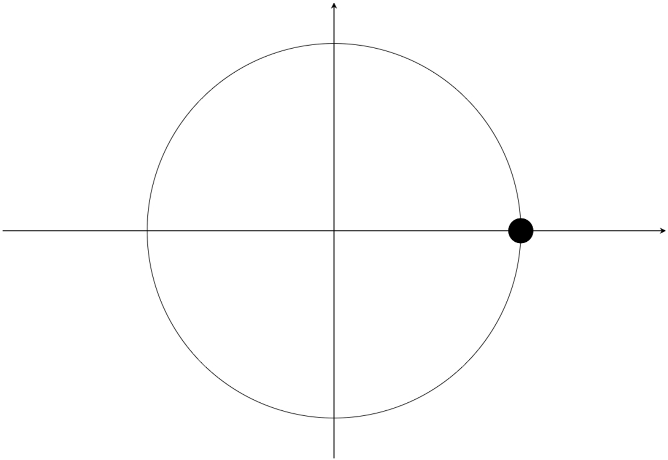
The $9 + 11 = 20$ division of the perfect fifth is in striking analogy to the $5 + 7 = 12$ division of the octave.

Stacking Major Thirds (in a Perfect Fifth)



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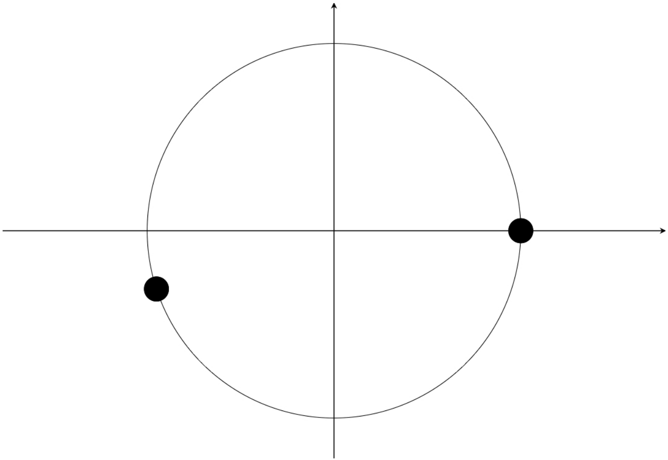

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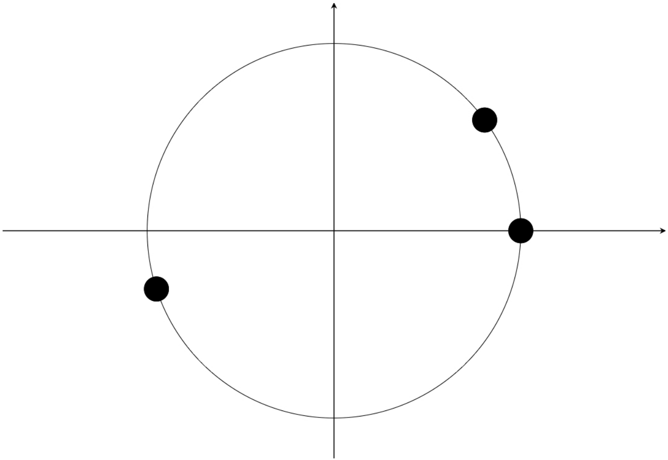

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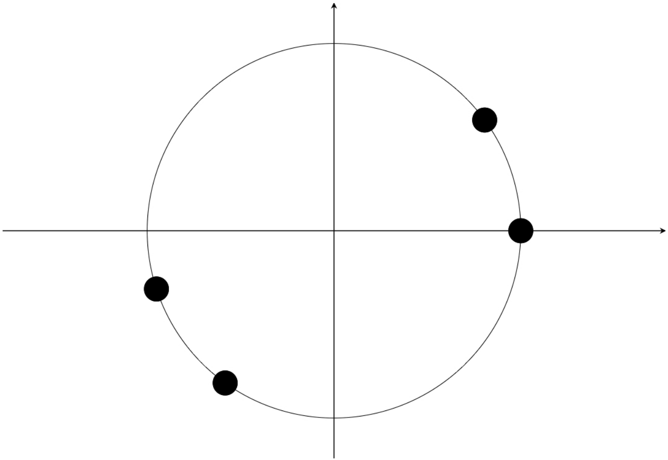

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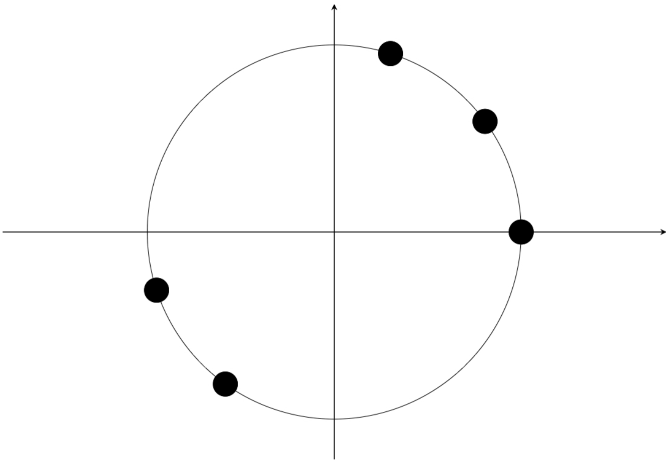

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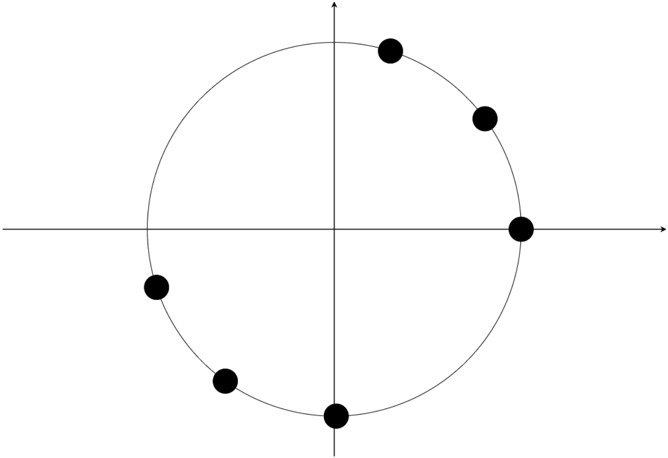

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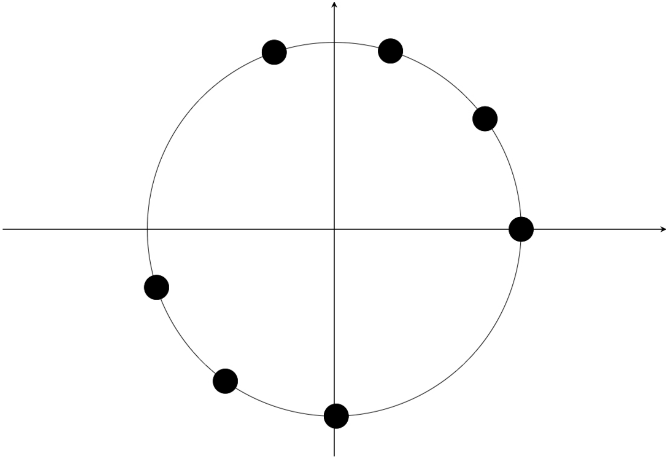

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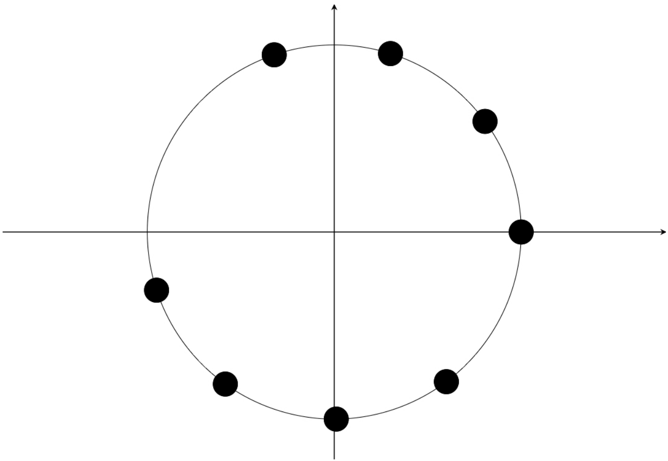

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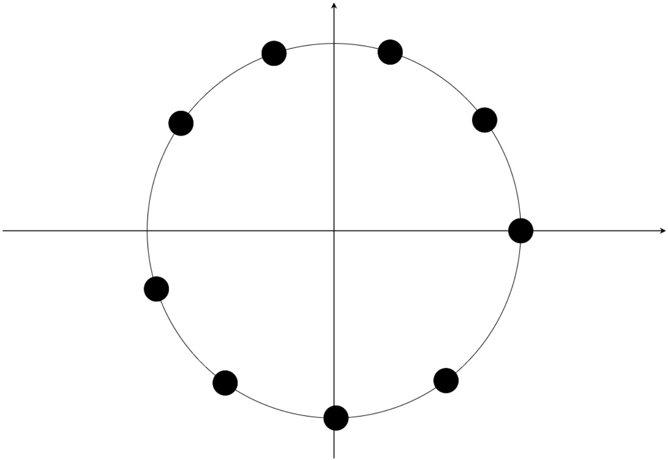

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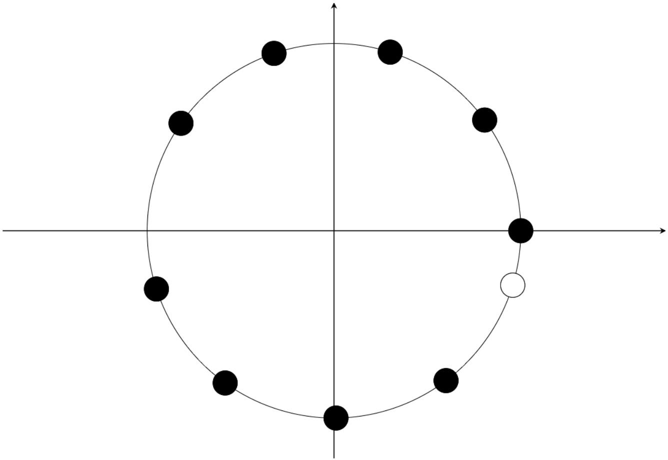

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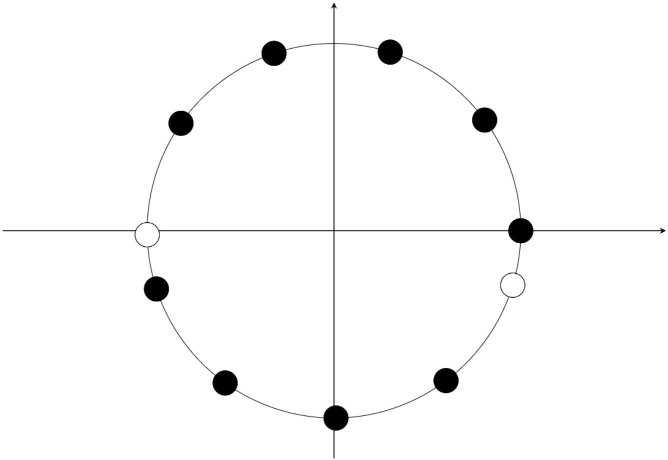

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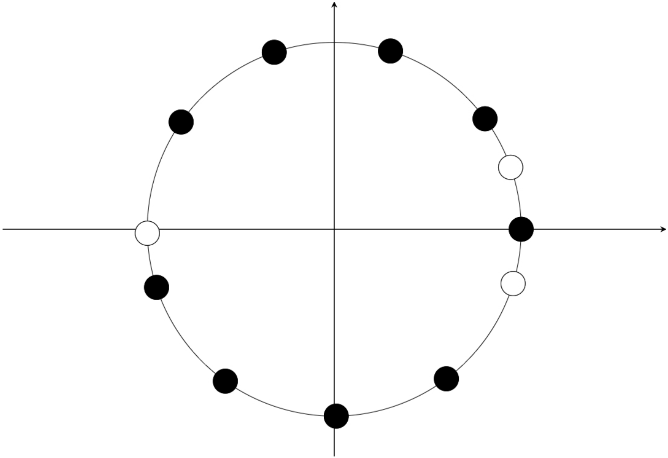

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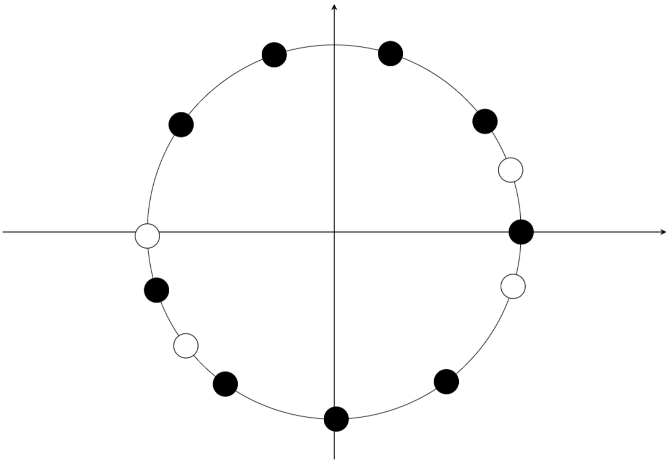

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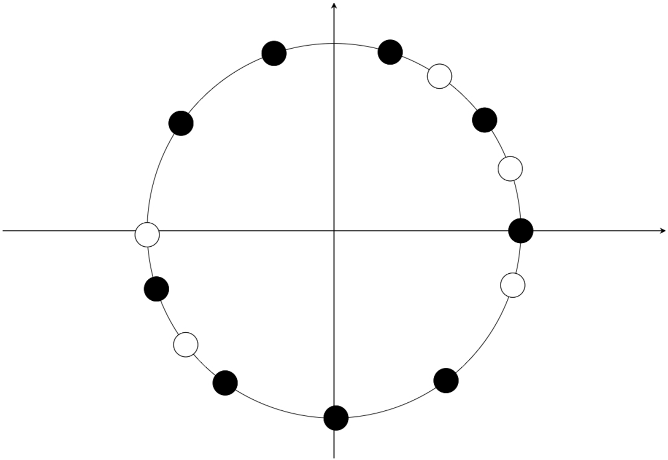

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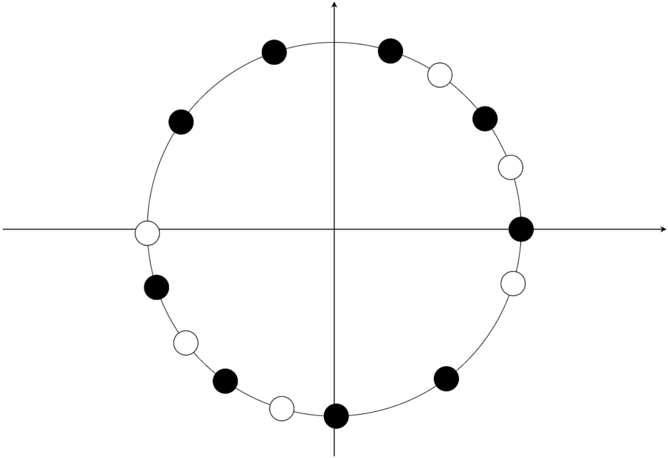

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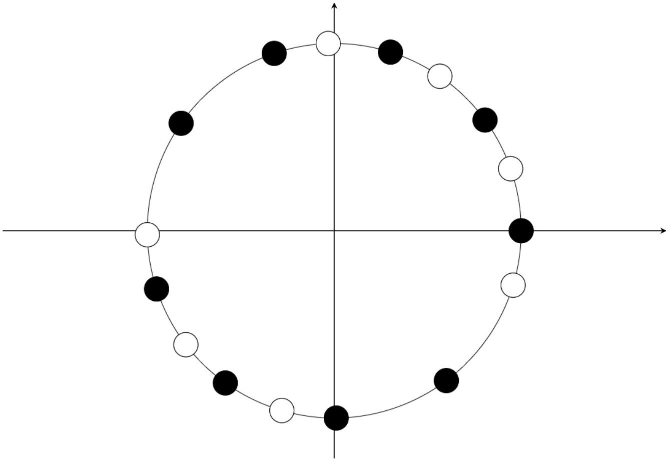

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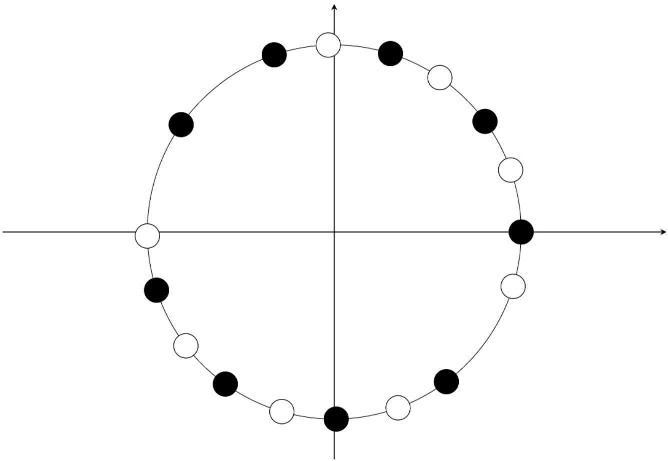

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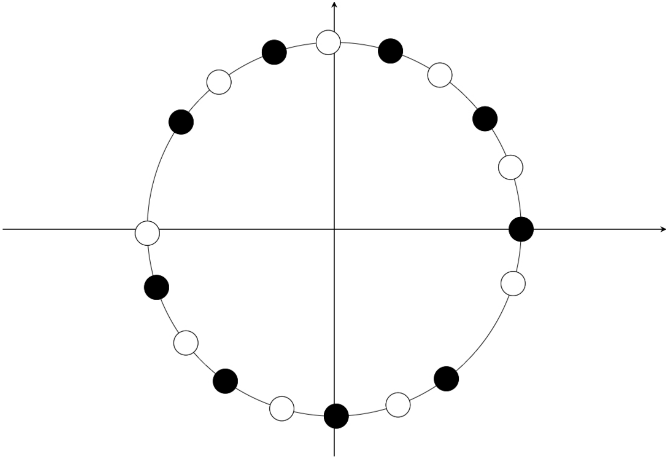

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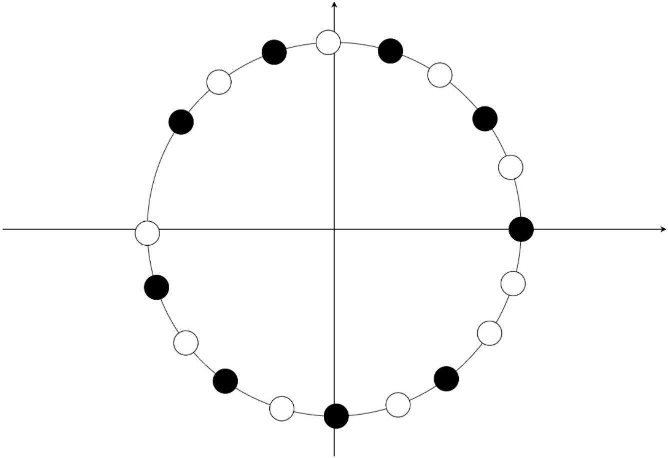

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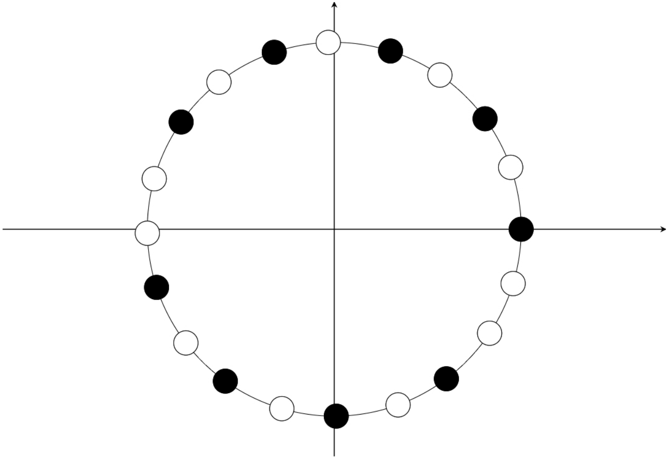

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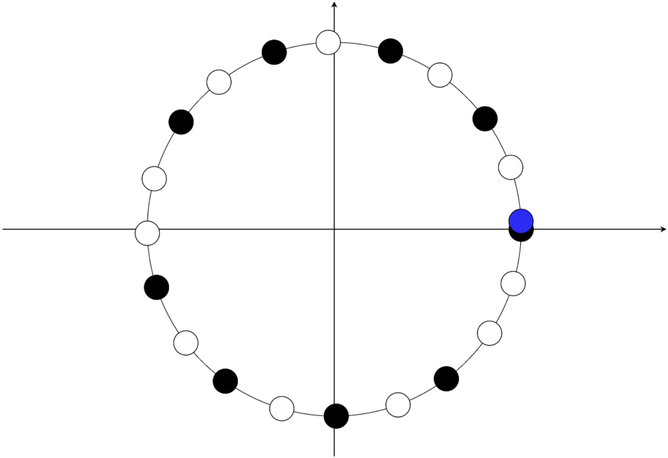
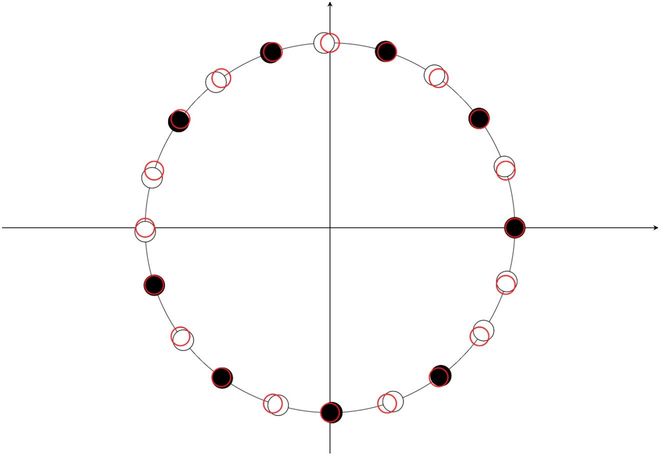
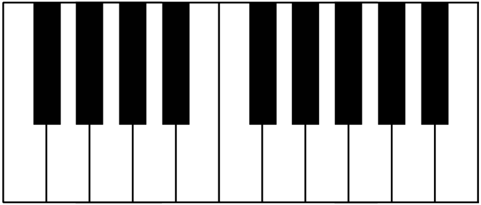


20-Tone Equal Temperment (Non-Octave Interval)



$1200 \cdot \log_2(3/2) = 701.955...$ cents $=$ whole interval
so notes are $\gamma = 35.097...$ cents apart.

One Perfect Fifth on a γ-Keyboard



We know exactly where the 9 (black keys) and 20 (total keys) come from.

Combinatorics of 20-Tone Chromatic Scale

$$\log_{3/2}(5/4) = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{6 + \cdots}}}}}$$

Combinatorics of 20-Tone Chromatic Scale

$$\log_{3/2}(5/4) = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{6 + \cdots}}}}}$$

$$\frac{1}{2} < \frac{11}{20} < \frac{82}{149} < \ldots < \log_{3/2}(5/4) < \ldots < \frac{71}{129} < \frac{5}{9} < 1$$