The Jellyfish Algorithm

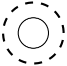
Ellie Grano

UC Santa Barbara

November 20, 2010

Background

- Evaluation algorithms common in topology
  - Examples: Kauffman bracket, HOMFLY polynomial
  - Idea:

 $\rightarrow$ number

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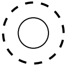

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- 2008 - Morrison, Peters, and Snyder
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    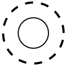 $\to$ number

- 2008 - Morrison, Peters, and Snyder
  Skein theory for the D_{2n} planar algebra

- 2009 - Bigelow
  Skein theory for the ADE planar algebras

  ”jellyfish algorithm” introduced

The Temperley-Lieb planar algebra

- For each k, $\mathcal{TL}_{2k}$ is an algebra over $\mathbb{C}(q)$. As a vector space, $\mathcal{TL}_{2k}$ is spanned by diagrams with k nonintersecting strands. The multiplication operation is vertical stacking. We also have the following ”bubble bursting” relation:

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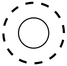
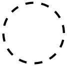
   $= \delta \cdot$  , where $\delta = q + q^{-1}$

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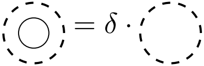

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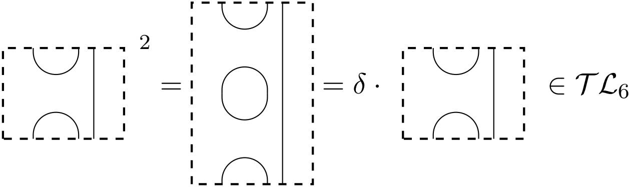

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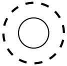
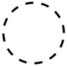
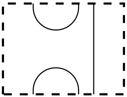
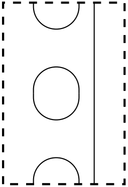
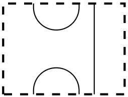
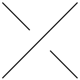
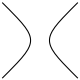
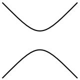
 $= \delta \cdot$ , where $\delta = q + q^{-1}$

 $^2 =$  $= \delta \cdot$  $\in \mathcal{TL}_6$

- These vector spaces assemble together into a planar algebra with $\mathcal{TL}_0 \cong \mathbb{C}(q)$.

The Temperley-Lieb planar algebra

-  $:= iq^{\frac{1}{2}}$  $- iq^{-\frac{1}{2}}$  $\in \mathcal{TL}_4$

  This satisfies R2 and R3.
  For R1, we get a positive twist factor $iq^{3/2}$.

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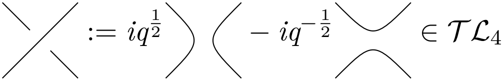
- 

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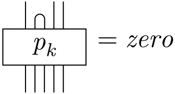
- Jones-Wenzl projections.
  For each k there is a unique element $p_k \in \mathcal{TL}_{2k}$ such that:
  - $p_k^2 = p_k$
  - p_k is uncappable.

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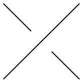
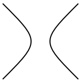
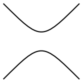
-  $:= iq^{\frac{1}{2}}$  $- iq^{-\frac{1}{2}}$  $\in \mathcal{TL}_4$

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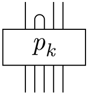
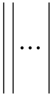
    

    $= zero$

  It follows that:

  $p_k =$ 

  $+ \sum \alpha_Q \cdot Q$, where each Q contains a cap.

Temperley-Lieb when q is a root of unity

If $q = e^{i\pi/n+1}$, then p_n becomes negligible. So for $\mathcal{TL}$ at this value of q, we must add the relation $p_n = zero$ (this gives us the A_n planar algebra). For example, if $q = e^{i\pi/6}$, then $\mathcal{TL}$ will have the relations:

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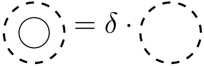
- 

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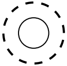
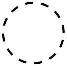
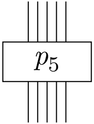
-  $= \delta \cdot$ 
- 

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The D_{2n} planar algebra ($\mathcal{P}$)

Fix $n = 2$ and $q = e^{i\pi/6}$. Define $\mathcal{P}$ to be the planar algebra generated by a single S-box in $\mathcal{P}_4$ subject to the following relations:

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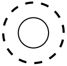
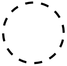
-  $= \delta \cdot$ 

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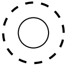
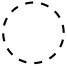
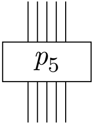
-  $= \delta \cdot$ 
-  $= zero$

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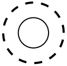
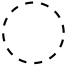
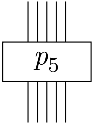
-  $= \delta \cdot$ 
- 

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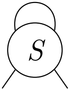
- 

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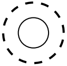
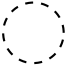
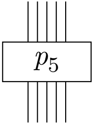
-  $= \delta \cdot$ 
- 

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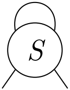
- 

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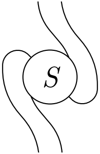
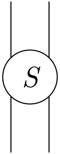
- 

$= i \cdot$ 

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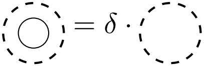
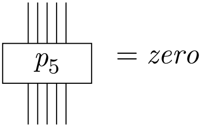
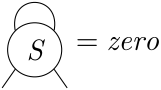
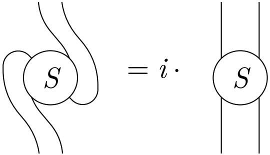
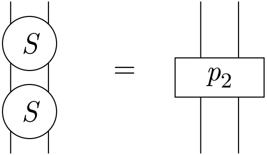
- 

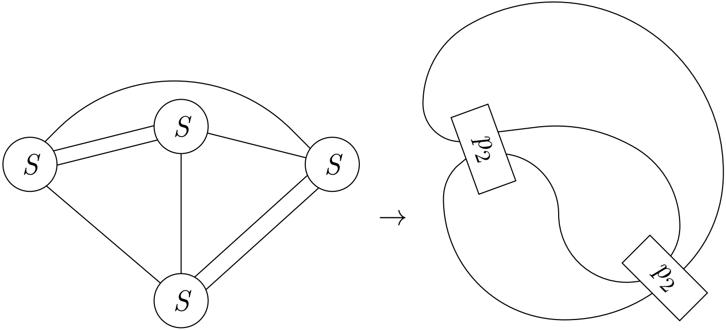
- 

- 

- 

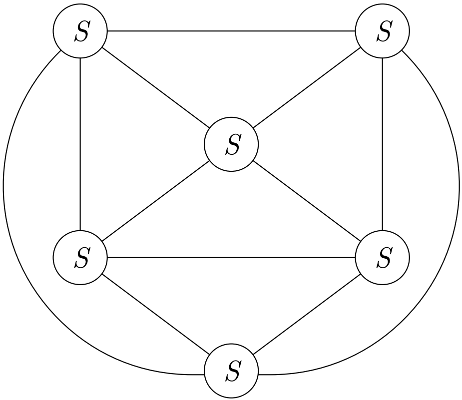
- 

An example of evaluating a diagram using the relations:



$\in \mathcal{T}\mathcal{L}_0$

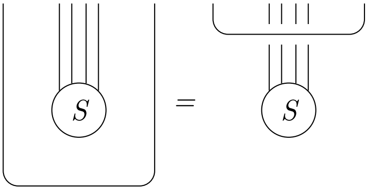
But what about:



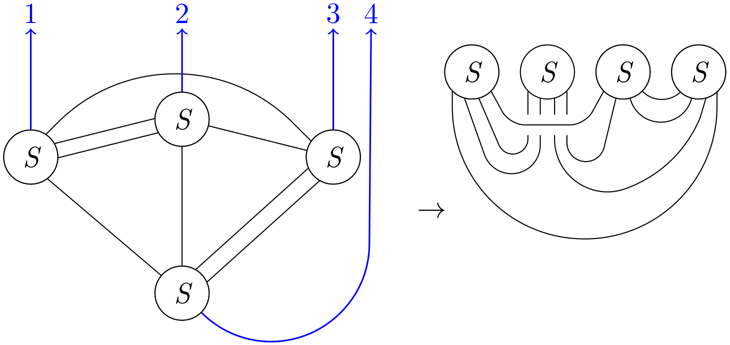
Partial braiding

Theorem

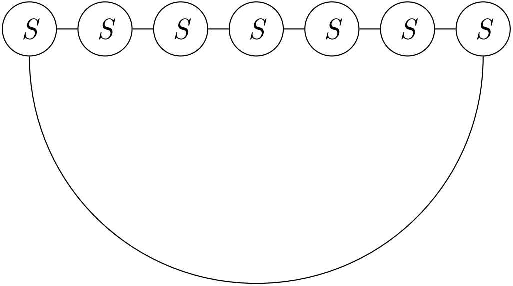
The relations imply the following partial braiding:



Example of the jellyfish algortihm



Will there always be an S^2?



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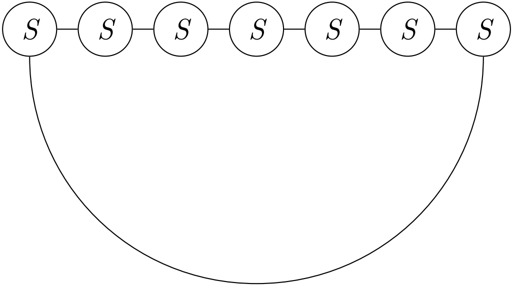

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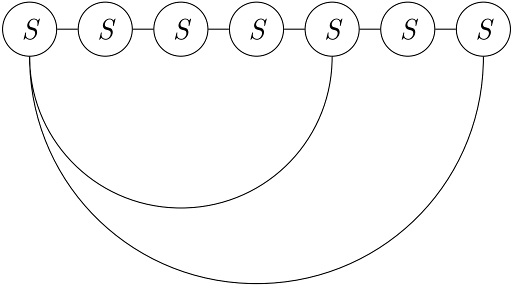

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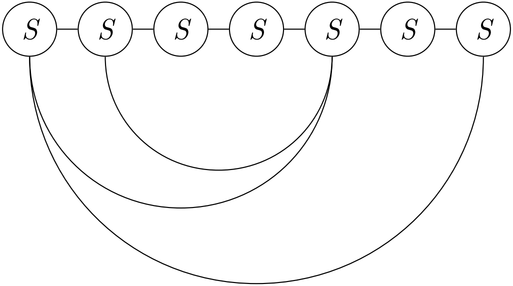

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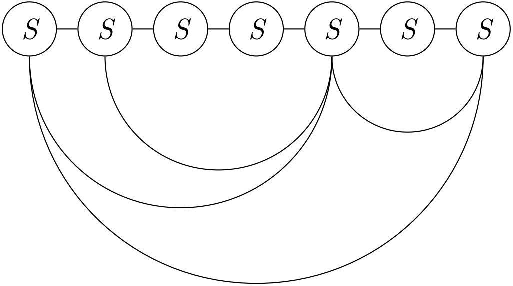

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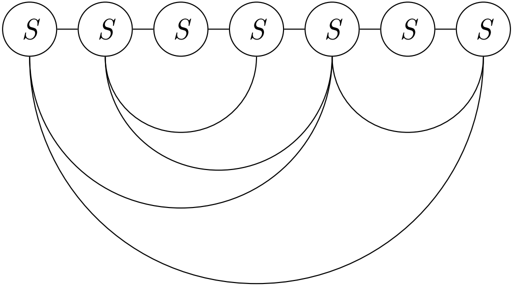

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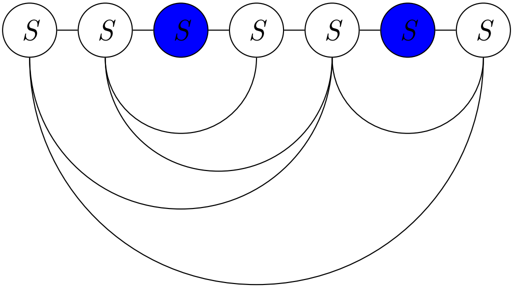

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Part I

- Draw an arc for each S-box
- Order the arcs
- Drag the S-boxes in order under any strands
- Evaluate crossings
- Go to Part II

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Part II

- If there are zero S-boxes, evaluate as in $\mathcal{TL}$
- If there is a cap on an S-box, evaluate as zero
- If there are two or more S-boxes
  - Pick a pair of S-boxes connected by at least two strands
  - Choose two strands connecting the pair and replace with a p_2
  - Put in the correct coefficient
  - Start Part II again

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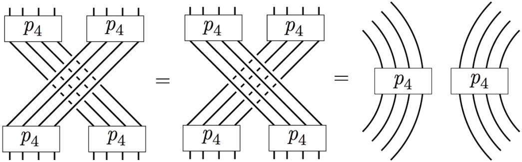
- If there is a cap on an S-box, evaluate as zero
- If there are two or more S-boxes
  - Pick a pair of S-boxes connected by at least two strands
  - Choose two strands connecting the pair and replace with a p_2
  - Put in the correct coefficient
  - Start Part II again

Change the ordering of the arcs

Fact (for our value of q):



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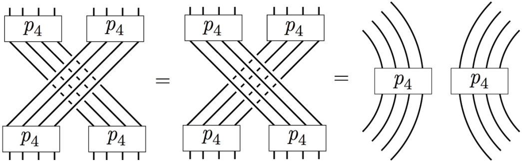

Idea: Add a p_4 along an S-box arc. Do this for each arc crossing that needs to be changed, then use the above fact.

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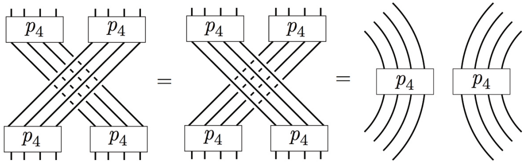

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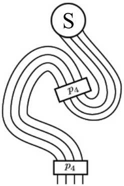
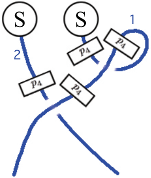
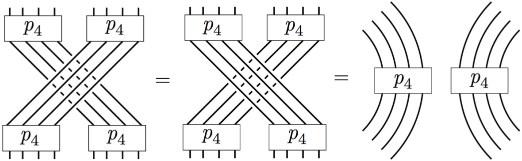


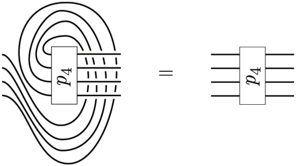


Alter the path of an arc

Notice that



implies:



Thus we have all three Reidemeister moves for the S-box arcs.

Check the algorithm respects the relations

Fix $n = 2$ and $q = e^{i\pi/6}$. Define $\mathcal{P}$ to be the planar algebra generated by a single S-box in $\mathcal{P}_4$ subject to the following relations:

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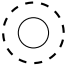
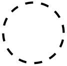
-  $= \delta \cdot$ 

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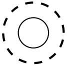
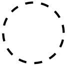
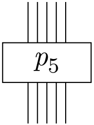
-  $= \delta \cdot$ 
- 

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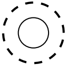
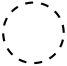
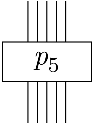
-  $= \delta \cdot$ 
- 

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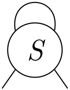
- 

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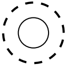
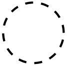
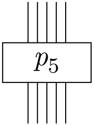
-  $= \delta \cdot$ 
- 

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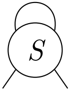
- 

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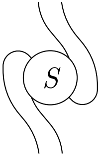
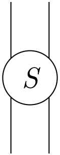
- 

$= i \cdot$ 

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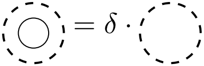
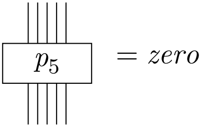
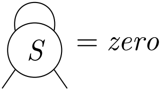
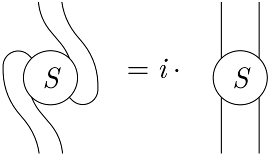
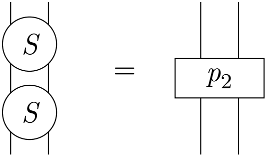
- 

- 

- 

- 

- 

Summary

We have just proved the following:

Theorem

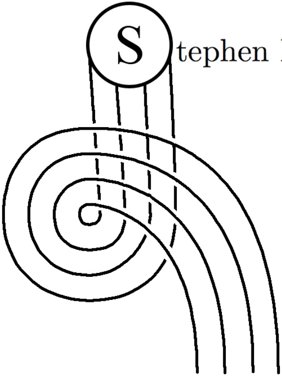
The defined planar algebra is not trivial

This is part of the **Kuperberg program**:

Give a presentation for every interesting planar algebra, and prove as much as possible about the planar algebra using only its presentation.

Thank You

I also want to thank my advisor



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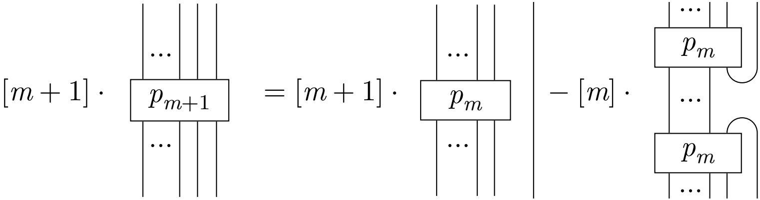
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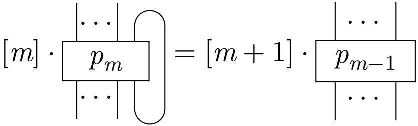
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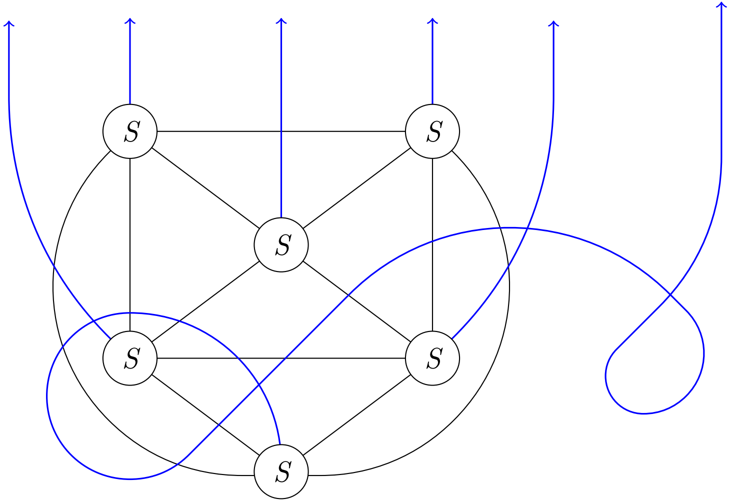
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