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On K3 surfaces with large Picard number

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A *K3 surface* is a simply connected compact complex manifold of dimension two with a nowhere-vanishing holomorphic 2-form. K3 surfaces have received much attention in the last 25 years, both because of the important place they occupy in the classification of compact complex surfaces [4], and because the "period map" for K3 surfaces is quite well-behaved (there is a "global Torelli theorem" [15, 1, 17, 7], and a "surjectivity of the period map" theorem [5, 21, 6, 19, 9].) Among classical examples of K3 surfaces are the "Kummer surfaces", which play a crucial role in the theory, and which are defined as follows. Let Z be a complex torus of dimension two, ι be an involution on Z induced by multiplication by -1 on the universal cover $\mathbb{C}^2$, and Y be the minimal resolution of singularities of Z/ι. Then Y is a *Kummer surface;* all Kummer surfaces are K3 surfaces.

The *Picard number* of a K3 surface is the rank of its group of line bundles; this rank ranges from 0 to 20. If X is a K3 surface with Picard number 20, then Shioda and Inose [18] have constructed an involution ι on X such that the quotient X/ι is birational to a Kummer surface. This gives rise to a diagram

$$\begin{tikzcd} X \arrow[dr, dashed] & & Z \arrow[dl, dashed] \\ & Y & \end{tikzcd}$$

in which the dotted arrows are rational maps of degree 2, X and Y are K3 surfaces, and Z is a complex torus. Shioda and Inose further show that this diagram induces an isomorphism of integral Hodge structures on the transcendental lattices of X and Z.

The main result of this paper is a generalization of the construction of Shioda and Inose. We show that there is a diagram analogous to theirs for any algebraic K3 surface of Picard number 19 or 20, and give precise conditions

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for the existence of such a diagram when the Picard number is 17 or 18. (Such a diagram cannot exist for algebraic K3 surfaces of Picard number less than 17.) This work was prompted by a remark of Takayuki Oda [13], who conjectured that analogues of the Shioda-Inose construction should exist when the Picard number is 17, 18, or 19.

The plan of the paper is as follows: Section 1 reviews definitions and results about Hodge theory, complex tori, and K3 surfaces. Section 2 summarizes the work of Nikulin [12] on embeddings of quadratic forms, and draws some consequences for K3 surfaces and complex tori. Sections 3, 4, and 5 discuss involutions on K3 surfaces and complex tori, relying heavily on two other papers of Nikulin [10, 11]. In Sect. 6, we generalize the Shioda-Inose construction, and in Sect. 7 we discuss Oda's conjecture.

1. Hodge structures, complex tori, and K3 surfaces

Definition 1.1. A *lattice* is a free $\mathbb{Z}$-module of finite rank equipped with a $\mathbb{Z}$-valued symmetric bilinear form $b(x,y)$. If L_1 and L_2 are two lattices, then $L_1 \oplus L_2$ denotes the orthogonal direct sum of L_1 and L_2; L^n denotes

$$L^n = L \oplus L \oplus \ldots \oplus L \quad (n \text{ factors}).$$

If L is a lattice and m is an integer, then $L(m)$ denotes the same $\mathbb{Z}$-module with a form which has been altered by multiplication by m, that is,

$$b_{L(m)}(x,y) = m(b_L(x,y)).$$

An isomorphism of lattices preserving the bilinear form is called an *isometry*; note that L is not isometric to $L(m)$ when $|m|>1$. The group of self-isometries of a lattice L is denoted by $O(L)$.

A lattice is *even* if the associated quadratic form takes on only even values, and is *odd* if the quadratic form takes on some odd value. The *discriminant* of a lattice L, written $\operatorname{discr}(L)$, is the determinant of the matrix of its bilinear form. A lattice is *non-degenerate* if its discriminant is non-zero, and *unimodular* if its discriminant is ± 1. If L is a non-degenerate lattice, the *signature* of L is a pair $(s_{(+)}, s_{(-)})$, where $s_{(\pm)}$ denotes the multiplicity of the eigenvalue ± 1 for the quadratic form on $L \otimes \mathbb{R}$. (Note that with this definition, the statement that a given lattice has a signature $(s_{(+)}, s_{(-)})$ automatically implies that L is non-degenerate.) A lattice is *indefinite* if the associated quadratic form takes on both positive and negative values; in the non-degenerate case, this is true if and only if $\min(s_{(+)}, s_{(-)}) > 0$.

Examples 1.2. (i) U denotes the *hyperbolic plane*, that is, U is a free $\mathbb{Z}$-module of rank 2 whose bilinear form has matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This is clearly an even lattice; note that $U(-m) \cong U(m)$ for any m.

(ii) E_8 denotes the unique even unimodular positive definite lattice of rank 8; the bilinear form on E_8 is given by the matrix

$$\begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & -1 & & & \\ & & -1 & 2 & 0 & & & \\ & & -1 & 0 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{bmatrix}$$

(iii) $\langle k\rangle$ denotes the lattice of rank 1 such that $b(x,x)=k$ for any generator x of $\langle k\rangle$.

Theorem 1.3 (Milnor [8]). *Let L be an indefinite unimodular lattice. If L is odd, then*

$$L\cong\langle 1\rangle^m\oplus\langle -1\rangle^n$$

for some m and n. If L is even, then

$$L\cong U^m\oplus E_8(\pm 1)^n$$

for some m and n. In particular, the signature and parity of L determine L up to isometry.

If X is a compact Kähler surface with the property that $H^2(X,\mathbf{Z})$ is torsion-free, then the *intersection pairing* gives $H^2(X,\mathbf{Z})$ the structure of a lattice. This lattice is unimodular by Poincaré duality, so that $H^2(X,\mathbf{Z})$ is determined by its signature and parity. The Hodge index theorem [2] says that the signature of the lattice $H^2(X,\mathbf{Z})$ is $(2h^{2,0}+1,\ h^{1,1}-1)$, where $h^{i,j}=\dim H^{i,j}(X)$.

Definition 1.4. Let L be a lattice. A *Hodge structure of weight 2* on L consists of a "Hodge decomposition"

$$L\otimes\mathbf{C}=L^{2,0}\oplus L^{1,1}\oplus L^{0,2}$$

such that $\overline{L^{2,0}}=L^{0,2}$, and $\overline{L^{1,1}}=L^{1,1}$ (where $\overline{}$ denotes complex conjugation), and such that

$$\begin{aligned} &b(x,\bar{x})>0 && \text{for } 0\neq x\in L^{2,0}, \\ &b(x,y)=0 && \text{for } x,y\in L^{2,0}, \text{ and} \\ &b(x,y)=0 && \text{if } x\in L^{2,0}\oplus L^{0,2} \text{ and } y\in L^{1,1}. \end{aligned}$$

A *Hodge isometry* is an isometry $\phi\colon L_1\xrightarrow{\sim}L_2$ between lattices with Hodge structures which preserves the Hodge decompositions.

A *signed Hodge structure (of weight two)* on a lattice L consists of a Hodge structure on L such that the quadratic form restricted to $L^{1,1}\cap(L\otimes\mathbf{R})$ has signature $(1,n-1)$, together with a choice of one of the (two) components of

$$\{x\in L^{1,1}\cap(L\otimes\mathbf{R}):b(x,x)>0\}. \qquad (*)$$

A *signed Hodge isometry* is a Hodge isometry between two lattices with signed Hodge structures which preserves the choice of component of (*).

A *polarized Hodge structure (of weight two)* is a Hodge structure with the property that the quadratic form is negative-definite when restricted to $L^{1,1}\cap(L\otimes\mathbb{R})$.

Let X be a compact Kähler surface such that $H^2(X,\mathbb{Z})$ is torsion-free. The lattice $H^2(X,\mathbb{Z})$ has a natural signed Hodge structure of weight two: we take the usual Hodge decomposition

$$H^2(X,\mathbb{C})\cong H^{2,0}(X)\oplus H^{1,1}(X)\oplus H^{0,2}(X)$$

together with the component of

$$\{x\in H^{1,1}(X,\mathbb{R}): b(x,x)>0\}$$

which contains the cohomology class of any Kähler metric. The Hodge index theorem [2] guarantees that the signature of the form on $H^{1,1}(X,\mathbb{R})$ is $(1,h^{1,1}-1)$.

Let $NS(X)$ be the Néron-Severi group of X, that is, the group of line bundles on X, modulo those algebraically equivalent to zero. $NS(X)$ has a natural embedding in $H^2(X,\mathbb{Z})$, and can be identified with $H^2(X,\mathbb{Z})\cap H^{1,1}(X)$, giving $NS(X)$ the structure of a lattice. The *Picard number of* X, denoted by $\rho(X)$, is the rank of $NS(X)$. The *transcendental lattice of* X, denoted by T_X, is the orthogonal complement of $NS(X)$ in $H^2(X,\mathbb{Z})$. The lattice T_X inherits a Hodge structure from the one on $H^2(X,\mathbb{Z})$.

If the Hodge structure on T_X is polarized, then its signature must be $(2h^{2,0}, h^{1,1}-\rho+1)$ so that the signature of the lattice $NS(X)$ is $(1,\rho-1)$. But then $NS(X)$ contains an element of positive square-length; by a theorem of Kodaira [4; Theorem 8], this implies that X is algebraic. Conversely, if X is algebraic, then the signature of $NS(X)$ is $(1,\rho-1)$, which implies that the Hodge structure on T_X is polarized.

Let $\chi_{\mathrm{top}}(X)$ denote the topological Euler characteristic of X.

Theorem-Definition 1.5 (Kodaira [4; Sect. 6]). Let X be a compact Kähler surface with trivial canonical bundle. Then $h^{2,0}(X)=1$, and either

(i) $X=\mathbb{C}^2/L$ is a complex torus of (complex) dimension 2; in this case, $h^{1,0}(X)=2$ and $\chi_{\mathrm{top}}(X)=0$, or

(ii) X is a *K3 surface*, that is, $h^{1,0}(X)=0$ and $\chi_{\mathrm{top}}(X)=24$. (In fact, a K3 surface can be defined as a compact complex surface with trivial canonical bundle such that $h^{1,0}(X)=0$, but Siu [20] has recently shown that every K3 surface is Kähler.)

If X is a complex torus, then it is easy to see directly that $H^2(X,\mathbb{Z})\cong U^3$. In particular, $H^2(X,\mathbb{Z})$ is torsion-free, so that $H^2(X,\mathbb{Z})$ and T_X carry natural Hodge structures. X is algebraic when T_X is polarized; in this case, we call X an *abelian surface*.

If X is a K3 surface, then $H^2(X,\mathbb{Z})$ has no torsion [16; Chap. IX, Sect. 3]. Thus, $H^2(X,\mathbb{Z})$ and T_X carry natural Hodge structures. Moreover, a computation involving the Wu formula [22] shows that $H^2(X,\mathbb{Z})$ is an even lattice (cf. [8] or [16]). The signature of this lattice is (3, 19) by the Hodge index theorem, so Theorem 1.3 implies that $H^2(X,\mathbb{Z})$ is isometric to the *K3 lattice* $\Lambda = U^3 \oplus E_8(-1)^2$.

For complex tori and K3 surfaces, the following results go by the name "the surjectivity of the period mapping":

Theorem 1.6 (Shioda [17]). *Given a signed Hodge structure on U^3, there exists a complex torus X of dimension two and a signed Hodge isometry*

$$\phi\colon H^2(X,\mathbb{Z}) \xrightarrow{\sim} U^3$$

(with respect to the given signed Hodge structure).

Theorem 1.7 ([5, 21, 6, 19, 9]). *Given a signed Hodge structure on the K3 lattice Λ, there exists a K3 surface X and a signed Hodge isometry*

$$\phi\colon H^2(X,\mathbb{Z}) \xrightarrow{\sim} \Lambda$$

(with respect to the given signed Hodge structure).

Definition 1.8. An embedding $M \hookrightarrow L$ of lattices is *primitive* if L/M is free. Two primitive embeddings $M \hookrightarrow L$, $M \hookrightarrow L'$ are *isomorphic* if there is an isometry $L \xrightarrow{\sim} L'$ which induces the identity map on M.

Corollary 1.9. *Let Λ be the K3 lattice.*

(i) *Suppose $S \hookrightarrow U^3$ (resp. $S \hookrightarrow \Lambda$) is a primitive sublattice of signature $(1, \rho-1)$. Then there exists an abelian surface (resp. algebraic K3 surface) X and an isometry $NS(X) \xrightarrow{\sim} S$.*

(ii) *Suppose $T \hookrightarrow U^3$ (resp. $T \hookrightarrow \Lambda$) is a primitive sublattice of signature $(2, 4-\rho)$ (resp. $(2, 20-\rho)$). Then there exists an abelian surface (resp. algebraic K3 surface) X and an isometry $T_X \xrightarrow{\sim} T$.*

Proof. Let L denote U^3 (resp. Λ), and let $b(x, y)$ denote the bilinear form on L.

(i) Choose a subspace $\Sigma \subset L\otimes\mathbb{R}$ such that $\Sigma \cap L = S$, and $b|_\Sigma$ has signature (1, 3) (resp. signature (1, 19)). Pick some non-zero $\omega \in L\otimes\mathbb{C}$ such that $\omega \perp \Sigma$ and $b(\omega,\omega)=0$. Define

$$L^{2,0} = \mathbb{C}\omega; \quad L^{1,1} = \Sigma\otimes\mathbb{C}; \quad L^{0,2} = \mathbb{C}\bar{\omega}.$$

Then $L\otimes\mathbb{C} = L^{2,0} \oplus L^{1,1} \oplus L^{0,2}$ is a Hodge decomposition; choosing either component of

$$\{x \in L^{1,1} \cap (L\otimes\mathbb{R}) : b(x,x) > 0\},$$

makes this into a signed Hodge structure. By Theorems 1.6 and 1.7, there is a complex torus (resp. K3 surface) X and a signed Hodge isometry

$$\phi\colon H^2(X,\mathbb{Z}) \xrightarrow{\sim} L.$$

But now $\phi|_{NS(X)}$ gives an isometry of $NS(X)$ with

$$L^{1,1} \cap L = \Sigma \cap L = S.$$

Since $NS(X)$ has signature $(1, \rho-1)$, X is algebraic.

(ii) Let S be the orthogonal complement of T in L, and apply part (i): we get

$$T_X = NS(X)^\perp \xrightarrow{\sim} S^\perp = T. \quad \text{Q.E.D.}$$

2. Discriminant-forms and embeddings of lattices

Definition 2.1. Let A be a finite abelian group. The *length of* A, denoted $l(A)$, is the minimum number of generators of A. A *quadratic form on* A is a map

$$q: A \to \mathbb{Q}/2\mathbb{Z}$$

together with a symmetric bilinear form

$$b: A \times A \to \mathbb{Q}/\mathbb{Z}$$

such that

1) $q(na) = n^2 q(a)$ for all $n \in \mathbb{Z}$ and $a \in A$.

2) $q(a+a') - q(a) - q(a') \equiv 2b(a, a') (\mathrm{mod}\, 2\mathbb{Z})$. Note that if q is a quadratic form on A, then so is $-q$ (with bilinear form $-b$).

If L is a non-degenerate even lattice, then there is a natural embedding

$$L \hookrightarrow L^* = \mathrm{Hom}(L, \mathbb{Z}).$$

The ($\mathbb{Q}$-valued) quadratic form on L^* induces a quadratic form q_L, called the *discriminant-form of* L, on the finite abelian group $A_L = L^*/L$. Notice that $q_{L(-1)} \cong -q_L$, and that $q_{L \oplus M} \cong q_L \oplus q_M$. Given a non-degenerate even lattice L, we attach as invariants $(s_{(+)}, s_{(-)}, q_L)$, where $(s_{(+)}, s_{(-)})$ is the signature of L. The usefulness of these invariants is shown by the following

Theorem 2.2 (Kneser [3], Nikulin [12; Corollary 1.13.3]). *Let L be an even lattice with invariants $(s_{(+)}, s_{(-)}, q_L)$. Suppose that*

(i) $0 < s_{(+)}$
(ii) $0 < s_{(-)}$
(iii) $l(A_L) \leqq \mathrm{rank}(L) - 2$.

Then up to isometry, L is the only lattice with those invariants.

Lemma 2.3. *Let M_1 and M_2 be even lattices with the same signature and discriminant-form, and let L be an even lattice which is uniquely determined by its signature and discriminant-form. If there is a primitive embedding $M_1 \hookrightarrow L$, then there is a primitive embedding $M_2 \hookrightarrow L$.*

Proof. Let K be the orthogonal complement of M_1 in L. Then we have a chain of inclusions

$$(M_1 \oplus K) \subset L \subset L^* \subset (M_1 \oplus K)^*.$$

Since $A_{M_1 \oplus K} \cong A_{M_1} \oplus A_K$ (orthogonal direct sum), there is an isomorphism

$$\phi: A_{M_2 \oplus K} \xrightarrow{\sim} A_{M_1 \oplus K}$$

preserving the discriminant-forms. Define

$$L' = \{l \in (M_2 \oplus K)^* : \phi(l) \in L/(M_1 \oplus K)\}.$$

Then there is an embedding $M_2 \hookrightarrow L'$. If $m \in M_2^* \cap L'$, then $\phi(m) \in (M_1^* \cap L)/(M_1 \oplus K)$ so that $\phi(m) \in M_1$, since $M_1 \hookrightarrow L$ is primitive. Thus, $m \in M_2$ so that the embedding $M_2 \hookrightarrow L'$ is primitive.

Since ϕ preserves the discriminant-form, $q_{L'} \cong q_L$. Moreover, since $M_2 \oplus K \subset L'$ and $M_1 \oplus K \subset L$, L and L' have the same signature. Thus, $L \cong L'$. Q.E.D.

Another easy argument yields the following

Lemma 2.4 (Nikulin [12; Proposition 1.6.1]). *Let $M \hookrightarrow L$ be a primitive embedding of non-degenerate even lattices, and suppose that L is unimodular. Then*

$$q_{M^\perp} \cong -q_M.$$

Conversely, if M_1 and M_2 are non-degenerate even lattices which satisfy $q_{M_1} \cong -q_{M_2}$, then there is a primitive embedding of M_1 into an even unimodular lattice L such that $M_1^\perp \cong M_2$.

Corollary 2.5 (cf. [7; Theorem 2.4]). *Let T be a non-degenerate even lattice of rank r. Then there is a primitive embedding $T \hookrightarrow U^r$.*

Proof. By Lemma 2.4, since $q_{T(-1)} \cong -q_T$ there is an even unimodular lattice L and a primitive embedding $T \hookrightarrow L$ such that $T^\perp \cong T(-1)$. But then L has signature (r, r), so that $L \cong U^r$ by Theorem 1.3. Q.E.D.

Recall that by Corollary 1.9(ii), the possible transcendental lattices of abelian surfaces are all primitive sublattices $T \hookrightarrow U^3$ of signature $(2, 4-\rho)$.

Corollary 2.6. *Let T be an even lattice of signature $(2, k)$.*

(i) *If $k=0$ or 1, then there is a primitive embedding $T \hookrightarrow U^3$.*

(ii) *If $k=2$, then there is a primitive embedding $T \hookrightarrow U^3$ if and only if $T \cong U \oplus T'$.*

(iii) *If $k=3$, then there is a primitive embedding $T \hookrightarrow U^3$ if and only if $T \cong U^2 \oplus T'$.*

Proof. If $k=0$ or 1, then T has rank $\leqq 3$, so that $T \hookrightarrow U^3$ by Corollary 2.5.

If $k=2$ and $T \hookrightarrow U^3$, let $S = T^\perp$. Then $U \oplus S(-1)$ has the same signature as T; by Lemma 2.4, it also has the same discriminant-form. Since $l(A_T) = l(A_S) \leqq 4-2$, by Theorem 2.2, $T \cong U \oplus S(-1)$. Conversely, if $T \cong U \oplus T'$, then by Corollary 2.5, $T' \hookrightarrow U^2$; thus, $T \hookrightarrow U^3$.

If $k=3$ and $T \hookrightarrow U^3$, let $S = T^\perp$. Then $U^2 \oplus S(-1)$ has the same signature as T; by Lemma 2.4, it also has the same discriminant-form. Since $l(A_T) = l(A_S) \leqq 3-2$, by Theorem 2.2, $T \cong U^2 \oplus S(-1)$. Conversely, if $T \cong U^2 \oplus T'$, then by Corollary 2.5, $T' \hookrightarrow U$; thus, $T \hookrightarrow U^3$. Q.E.D.

We will need one further

Corollary 2.7. *Let $T_{\mathbb{Q}}$ be a sub-$\mathbb{Q}$-lattice of $U^3 \otimes \mathbb{Q}$ of signature* (2, 2). *Then the quadratic form of $T_{\mathbb{Q}}$ represents zero.*

Proof. Let $T = T_{\mathbb{Q}} \cap U^3$. By Corollary 2.6(ii), $T \cong U \oplus T'$. But U represents zero, hence $T_{\mathbb{Q}}$ also represents zero. Q.E.D.

The main result on embeddings of even lattices is

Theorem 2.8 (Nikulin [12; Theorem 1.14.4]). *Let M be an even lattice with invariants $(t_{(+)}, t_{(-)}, q_M)$, and let L be an even unimodular lattice of signature $(s_{(+)}, s_{(-)})$. Suppose that*

(i) $t_{(+)} < s_{(+)}$
(ii) $t_{(-)} < s_{(-)}$
(iii) $l(A_M) \leqq \operatorname{rank}(L) - \operatorname{rank}(M) - 2$.

Then there exists a unique primitive embedding of M into L.

This theorem has quite strong consequences for the structure of Néron-Severi groups and transcendental lattices of K3 surfaces. Let Λ be the K3 lattice.

Corollary 2.9. (i) *If $\rho \leqq 10$, then every even lattice S of signature $(1, \rho - 1)$ occurs as the Néron-Severi group of some algebraic K3 surface, and the primitive embedding $S \hookrightarrow \Lambda$ is unique.*

(ii) *The transcendental lattice of an algebraic K3 surface X with $\rho(X) \leqq 10$ is uniquely determined by its signature and discriminant-form.*

Corollary 2.10. (i) *If $12 \leqq \rho \leqq 20$, then every even lattice T of signature $(2, 20 - \rho)$ occurs as the transcendental lattice of some algebraic K3 surface, and the primitive embedding $T \hookrightarrow \Lambda$ is unique.*

(ii) *The Néron-Severi group of an algebraic K3 surface X with $12 \leqq \rho(X) \leqq 20$ is uniquely determined by its signature and discriminant-form.*

Remark 2.11. The case $\rho(X) = 11$ is missing from the two corollaries above. In fact, it follows from a stronger version of Theorem 2.2 proved by Nikulin ([12; Theorem 1.10.1]) that every even lattice of signature (1, 10) occurs as the Néron-Severi group of some algebraic K3 surface, and that every even lattice of signature (2, 9) occurs as the transcendental lattice of some algebraic K3 surface. However, no uniqueness results are known (without imposing additional hypotheses).

Proofs. (2.9) By Corollary 1.9, S occurs as the Néron-Severi group of some K3 surface if and only if there is a primitive embedding $S \hookrightarrow \Lambda$. Since $\rho \leqq 10$, we have

$$1 < 3; \quad \rho - 1 < 19; \quad l(A_S) \leqq \rho \leqq 22 - \rho - 2.$$

Thus, by Theorem 2.8, there exists a unique primitive embedding $S \hookrightarrow \Lambda$. Moreover, by Theorem 2.2 the orthogonal complement T of S (which is isomorphic to the transcendental lattice of any K3 surface whose Néron-Severi group is S) is uniquely determined by its signature and discriminant-form.

The proof of (2.10) is entirely analogous: the inequalities needed are

$$2<3; \quad 20-\rho<19; \quad l(A_T)\leqq 22-\rho\leqq 22-(22-\rho)-2$$

which hold for $12\leqq\rho\leqq 20$. Q.E.D.

3. Double covers and involutions

Let X be a compact complex surface. Let ι be an involution of X with isolated fixed points $Q_1,\dots,Q_k$, and let G be the group generated by ι. Let $\pi\colon X\to\bar{Y}$ be the quotient by G. $\bar{Y}$ has ordinary double points at the points $P_i=\pi(Q_i)$, so that if $\psi\colon Y\to\bar{Y}$ is the minimal resolution, then the exceptional divisors of ψ are smooth rational curves $C_i=\psi^{-1}(P_i)$ of self-intersection -2. We call the induced rational map $X \dashrightarrow Y$ the *rational quotient map.*

Let $\phi\colon Z\to X$ be the blowup at the points $Q_1,\dots,Q_k$ and let $E_i=\phi^{-1}(Q_i)$ be the exceptional divisors. Then the action of G on X lifts to an action of G on Z, and $Z/G\cong Y$:

$$\begin{tikzcd} Z \arrow[r, "\phi"] \arrow[d, "\tilde{\pi}"'] & X \arrow[d, "\pi"] \\ Y \arrow[r, "\psi"] & \bar{Y} \end{tikzcd}$$

Since $\tilde{\pi}$ is a double cover branched on the divisor $\sum_{i=1}^{k} C_i$, we get $\frac{1}{2}\sum_{i=1}^{k} C_i\in NS(Y)$.

Conversely, if $C_1,\dots,C_k$ are disjoint smooth irreducible rational curves on a surface with $\frac{1}{2}\sum_{i=1}^{k} C_i\in NS(Y)$, then there is a double cover $\tilde{\pi}\colon Z\to Y$ branched on $\frac{1}{2}\sum_{i=1}^{k} C_i$. $\tilde{\pi}^*(C_i)=2E_i$, and each E_i is an exceptional divisor of the first kind, so we may blowdown ΣE_i to recover the surface X.

Let H_Z be the orthogonal complement of $\{E_i\}$ in $H^2(Z,\mathbb{Z})$, and H_Y be the orthogonal complement of $\{C_i\}$ in $H^2(Y,\mathbb{Z})$. Then $H_Z\cong H^2(X,\mathbb{Z})$, and there are natural maps (cf. [18; Sect. 3])

$$\pi^*\colon H_Y\to H_Z\cong H^2(X,\mathbb{Z}); \qquad \pi_*\colon H^2(X,\mathbb{Z})\cong H_Z\to H_Y\subset H^2(Y,\mathbb{Z})$$

such that

$$\pi_*\pi^*(y)=2y; \qquad \pi^*\pi_*(x)=x+\iota^*(x); \qquad (y_1,y_2)=\tfrac{1}{2}(\pi^*y_1,\pi^*y_2).$$

Note also that

$$\pi^*(K_Y)=K_X.$$

Lemma 3.1

$$\pi_*(H^2(X,\mathbb{Z})^G)\cong H^2(X,\mathbb{Z})^G(2).$$

(In other words, π_* restricted to $H^2(X,\mathbb{Z})^G$ is an isomorphism onto its image which multiplies the intersection form by 2.)

Proof. If $x\in H^2(X,\mathbb{Z})^G$, then

$$\pi^*\pi_*(x)=x+\iota^*(x)=2x.$$

Thus, π_* restricted to $H^2(X,\mathbb{Z})^G$ is an isomorphism onto its image. If $x_1,x_2\in H^2(X,\mathbb{Z})^G$, then

$$(\pi_*x_1,\pi_*x_2)=\tfrac{1}{2}(\pi^*\pi_*x_1,\pi^*\pi_*x_2)=\tfrac{1}{2}(2x_1,2x_2)=2(x_1,x_2).\quad \text{Q.E.D.}$$

Proposition 3.2. *Suppose there is an even lattice $L\subset H^2(X,\mathbb{Z})^G$ with $L\cong U^n$. Let M be the orthogonal complement of $\pi_*(L)$ in $H^2(Y,\mathbb{Z})$ and suppose that* $\operatorname{discr}(M)=2^{2n}$. *Then $\pi_*(L)$ is a primitive sublattice of $H^2(Y,\mathbb{Z})$, and $\pi_*(L)\cong U(2)^n$.*

If in addition $T_X\subset L$, then π_* induces a Hodge isometry π_*: $T_X(2)\xrightarrow{\sim}T_Y$.

Proof. Note that $M^\perp$ is the minimal primitive sublattice of $H^2(Y,\mathbb{Z})$ containing $\pi_*(L)$, and that $\operatorname{discr}(M^\perp)=\operatorname{discr}(M)$ since $H^2(Y,\mathbb{Z})$ is unimodular. Thus,

$$2^{2n}=\operatorname{discr}(M^\perp)=\frac{\operatorname{discr}(\pi_*(L))}{[M^\perp:\pi_*(L)]^2}.$$

But $\pi_*(L)=U(2)^n$ by Lemma 3.1, so that $\operatorname{discr}(\pi_*(L))=2^{2n}$. Thus, $[M^\perp:\pi_*(L)]=1$, so that $\pi_*(L)$ is primitive.

If $T_X\subset L$, then π_* induces an isometry $T_X(2)\xrightarrow{\sim}T_Y$. Moreover,

$$\pi_*\colon H^2(X,\mathbb{Z})^G\to H^2(Y,\mathbb{Z})$$

preserves the Hodge decomposition, so that this is in fact a Hodge isometry. Q.E.D.

Lemma 3.3 (Nikulin [10; Lemma 3]). *Let $C_1,\dots,C_k$ be smooth irreducible disjoint rational curves on a K3 surface Y, and suppose $\frac{1}{2}\sum_{i=1}^{k}C_i\in NS(Y)$. Then $k=0$, 8, or 16.*

Proof. If $k>0$, let $\pi\colon X\dashrightarrow Y$ be the rational quotient map corresponding to the double cover branched on $\sum C_i$, and let $P_i\in X$ be the points corresponding to C_i. Then

$$\chi_{\text{top}}(X)=\chi_{\text{top}}(X-\{P_i\})+k=2\chi_{\text{top}}(Y-\{C_i\})+k=2(24-2k)+k.$$

Now $K_X=\pi^*(K_Y)\equiv 0$, so that by Theorem 1.5, X is a complex torus or a K3 surface. In the first case, $\chi_{\text{top}}(X)=0$ and $k=16$; in the second, $\chi_{\text{top}}(X)=24$ and $k=8$. Q.E.D.

4. Kummer surfaces

Definition 4.1. Let Z be a complex torus of dimension 2, and let ι be an involution on Z induced by multiplication by -1 on the universal cover $\mathbb{C}^2$. If $\pi\colon Z\dashrightarrow Y$ is the rational quotient by ι, then Y, which is a K3 surface, is called a

Kummer surface. ι has sixteen fixed points on Z, so that Y has sixteen exceptional curves. Note that Z is an abelian surface if and only if Y is an algebraic K3 surface.

Theorem 4.2 (Nikulin [10]). *There exists an even, negative-definite rank* 16 *lattice K, called the Kummer lattice, with the following properties:*

(i) $\operatorname{discr}(K) = 2^6$.

(ii) *If Y is a Kummer surface, then the minimal primitive sublattice of $H^2(Y, \mathbf{Z})$ containing the classes of the exceptional curves on Y is isomorphic to K.*

(iii) *K admits a unique primitive embedding into the K3 lattice Λ.*

(iv) *A K3 surface Y surface Y is a Kummer surface if and only if there is a primitive embedding $K \hookrightarrow NS(Y)$.*

The Kummer lattice was first described by D.B. Fuks (cf. [15; appendix to Sect. 5]).

Proposition 4.3. *Let Y be a Kummer surface, Z be the corresponding complex torus, $\pi: Z \dashrightarrow Y$ be the rational quotient map, and T_Y (resp. T_Z) be the transcendental lattice of Y (resp. Z). Then*

(i) (cf. Nikulin [10; Remark 2]) *π_* induces a Hodge isometry $T_Z(2) \cong T_Y$.*

(ii) $q_K \cong (q_{U(2)})^3$.

Proof. The Kummer involution acts as the identity on $H^2(Z, \mathbf{Z}) \cong U^3$; moreover, by (4.2)(ii), $\pi_*(H^2(Z, \mathbf{Z}))^\perp \cong K$ which has discriminant 2^6 by (4.2)(i). Statement (i) now follows immediately from Proposition 3.2, which also tells us that

$$K^\perp \cong \pi_*(H^2(Z, \mathbf{Z})) \cong U(2)^3$$

so that

$$q_K \cong -q_{K^\perp} \cong (q_{U(2)})^3$$

(since $-q_{U(2)} \cong q_{U(-2)} \cong q_{U(2)}$). Q.E.D.

Corollary 4.4. *Let Y be an algebraic K3 surface.*

(i) *If $\rho(Y) = 19$ or 20, then Y is a Kummer surface if and only if there is an even lattice T' with $T_Y \cong T'(2)$.*

(ii) *If $\rho(Y) = 18$, then Y is a Kummer surface if and only if there is an even lattice T' with $T_Y \cong U(2) \oplus T'(2)$.*

(iii) *If $\rho(Y) = 17$, then Y is a Kummer surface if and only if there is an even lattice T' with $T_Y \cong U(2)^2 \oplus T'(2)$.*

(iv) *If $\rho(Y) < 17$, then Y is not a Kummer surface.*

Proof. By (4.2)(iv), Y is a Kummer surface if and only if there is a primitive embedding $K \hookrightarrow NS(Y)$. Since K admits a unique primitive embedding into Λ by (4.2)(iii), such a primitive embedding $K \hookrightarrow NS(Y)$ exists if and only if there is a primitive embedding $T_Y \hookrightarrow K^\perp \cong U(2)^3$. But then there is some even lattice T'' with $T_Y \cong T''(2)$ and a primitive embedding $T'' \hookrightarrow U^3$. The corollary now follows from Corollary 2.6. Q.E.D.

Note that when $17 \leqq \rho \leqq 20$, every even lattice of the appropriate signature occurs as the transcendental lattice of a K3 surface, by Corollary 2.10. Corollary 4.4 thus shows that Kummer surfaces are rather rare among K3 surfaces with such Picard numbers.

5. Nikulin involutions

Definition 5.1. An involution ι on a K3 surface X is a *Nikulin involution* if $\iota^*(\omega)=\omega$ for every $\omega\in H^{2,0}(X)$.

Lemma 5.2 (Nikulin [11; Sect. 5]). (i) *Every Nikulin involution has eight isolated fixed points.*

(ii) *If $\pi\colon X \dashrightarrow Y$ is the rational quotient by a Nikulin involution, then Y is a K3 surface.*

Definition 5.3. The *Nikulin lattice* is an even lattice N of rank 8 generated by $\{c_i\}_{i=1}^8$ and $d=\frac{1}{2}\Sigma c_i$, with the bilinear form induced by

$$(c_i, c_j) = -2\delta_{ij}.$$

Lemma 5.4. (i) *The discriminant of N is 2^6.*

(ii) *If X is a K3 surface with a Nikulin involution ι, and $X \dashrightarrow Y$ is the rational quotient map, then the minimal primitive sublattice of $H^2(Y,\mathbf{Z})$ containing the classes of the exceptional curves on Y is isomorphic to N.*

Proof. (i) If N' is the sublattice of N spanned by $\{c_i\}$, then $\operatorname{discr}(N')=2^8$ and $[N:N']=2$, so that

$$\operatorname{discr}(N)=\frac{\operatorname{discr}(N')}{[N:N']^2}=2^6.$$

(ii) Let $C_1, \ldots, C_8$ be the exceptional divisors of $X \dashrightarrow Y$. Since $X \dashrightarrow Y$ is the rational quotient by an involution, $D=\frac{1}{2}\Sigma C_i \in NS(Y)$ as well, so that $N \hookrightarrow NS(Y) \subset H^2(Y,\mathbf{Z})$. To show that the embedding is primitive, suppose that $C=\Sigma m_i C_i \in NS(Y)$ with $m_i\in\mathbf{Q}$. Then $(C, C_j)=-2m_j\in\mathbf{Z}$, so that

$$C \equiv \sum_{i\in I} \tfrac{1}{2} C_i \pmod{N'}$$

for some $I\subset\{1,\ldots,8\}$. But by Lemma 3.3, $\#(I)=0$ or 8; if $\#(I)=0$, then $C\in N'$, while if $\#(I)=8$ then $C\equiv D \bmod N'$ so that $C\in N$. Q.E.D.

Definition 5.5. Let X be a K3 surface. The *Weyl group of X* is the subgroup

$$W(X) \subset \operatorname{Aut} H^2(X,\mathbf{Z})$$

generated by reflections in all elements of $NS(X)$ of square-length -2.

The following theorem of Nikulin is a consequence of the global Torelli theorem for K3 surfaces ([15, 1, 17, 7]):

Theorem 5.6 (Nikulin [11; Theorems 4.3, 4.7, 4.15]). *Let X be a K3 surface, let $G\cong\mathbf{Z}/2\mathbf{Z}$ be a subgroup of $O(H^2(X,\mathbf{Z}))$, and let $S_G=(H^2(X,\mathbf{Z})^G)^\perp$. Suppose that*

(i) *the lattice* S_G *is negative definite,*
(ii) *no element of* S_G *has square-length* -2, *and*
(iii) $S_G \subset NS(X)$.

Then there is a Nikulin involution ι on X and an element $w \in W(X)$ such that

$$\iota^* = w g w^{-1}$$

where g is the generator of G.

As a consequence, we get the following

Theorem 5.7. *Let* X *be a* $K3$ *surface such that* $E_8(-1)^2 \hookrightarrow NS(X)$. *Then there is a Nikulin involution* ι *on* X *such that if* $\pi\colon X \dashrightarrow Y$ *is the rational quotient map,*

(i) *there is a primitive embedding* $N \oplus E_8(-1) \hookrightarrow NS(Y)$,
(ii) π_* *induces a Hodge isometry* $T_X(2) \cong T_Y$,
(iii) $N \oplus E_8(-1)$ *has discriminant-form* $(q_{U(2)})^3$.

Proof. Let $\{c_j^i\}$ $(i=1,2;\ 1 \leqq j \leqq 8)$ be a basis of $E_8(-1)^2 = E_8(-1) \oplus E_8(-1)$ such that $c_j^1 \in E_8(-1) \oplus (0)$, $c_j^2 \in (0) \oplus E_8(-1)$, and for each fixed i, $\{c_j^i\}$ forms a basis of $E_8(-1)$ whose matrix is the negative of that in (1.2)(ii). Let $\phi\colon E_8(-1)^2 \hookrightarrow H^2(X, \mathbf{Z})$ be the embedding, and define an action of $G \cong \mathbf{Z}/2\mathbf{Z}$ on $H^2(X, \mathbf{Z})$ as follows: the generator $g \in G$ acts as

$$g(\phi(c_j^1)) = \phi(c_j^2); \quad g(\phi(c_j^2)) = \phi(c_j^1);$$
$$g(e) = e, \quad \text{for all } e \in \phi((E_8(-1)^2))^\perp.$$

(This is well-defined since the embedding ϕ is primitive, and $E_8(-1)^2$ is unimodular.) $S_G = (H^2(X, \mathbf{Z})^G)^\perp$ is generated by $\{\phi(c_j^1) - \phi(c_j^2)\}$, so that $S_G \subset \phi(E_8(-1)^2) \subset NS(X)$ and $S_G \cong E_8(-2)$. Since E_8 is an even, positive definite lattice, S_G is a negative-definite lattice which contains no element of square-length -2. Thus, by Theorem 5.6, there is a Nikulin involution ι on X and an element $w \in W(X)$ such that $\iota^* = w g w^{-1}$. For $x \in E_8(-1)^2$, let

$$\psi(x) = w(\phi(x)).$$

Then $\psi\colon E_8(-1)^2 \hookrightarrow NS(X) \subset H^2(X, \mathbf{Z})$ is another primitive embedding, since $W(X)$ preserves $NS(X)$. Moreover,

$$\iota^*(\psi(c_j^1)) = \psi(c_j^2); \quad \iota^*(\psi(c_j^2)) = \psi(c_j^1);$$
$$\iota^*(e) = e, \quad \text{for all } e \in \psi((E_8(-1)^2))^\perp.$$

Let $\pi\colon X \dashrightarrow Y$ be the rational quotient map. The minimal primitive lattice containing the exceptional divisors spans a copy of $N \hookrightarrow NS(Y)$. Moreover, the classes $\pi_*(\psi(c_1^1)), \ldots, \pi_*\psi((c_8^1))$ are orthogonal to N. Now by the formulas in Sect. 3,

$$\begin{aligned}
(((\pi_*\psi((c_j^1)), \pi_*\psi((c_k^1))) &= \tfrac{1}{2}(\pi^*\pi_*\psi((c_j^1)), \pi^*\pi_*\psi((c_k^1))) \\
&= \tfrac{1}{2}(\psi(c_j^1) + \iota^*\psi((c_j^1)), \psi(c_k^1) + \iota^*\psi((c_k^1))) \\
&= \tfrac{1}{2}(c_j^1, c_k^1) + \tfrac{1}{2}(c_j^2, c_k^2) \\
&= (c_j^1, c_k^1)
\end{aligned}$$

since $\{c_j^1\}$ and $\{c_j^2\}$ have identical bilinear form matrices. But this means that $\{\pi_* \psi((c_j^1))\}$ also spans a copy of $E_8(-1)$, so that $N \oplus E_8(-1) \hookrightarrow NS(Y)$. Since $E_8(-1)$ is unimodular and N is primitively embedded, the embedding $N \oplus E_8(-1) \hookrightarrow NS(Y)$ is also primitive.

Let L be the orthogonal complement of $\psi(E_8(-1)^2)$ in $H^2(X, \mathbf{Z})$. Then $L \cong U^3$. Since $L \hookrightarrow H^2(X, \mathbf{Z})^G$ and $\operatorname{discr}(N \oplus E_8) = 2^6$, by Proposition 3.2, π_* induces a Hodge isometry $T_X(2) \cong T_Y$, and $\pi_*(L)$ is a primitive sublattice of $H^2(Y, \mathbf{Z})$, isomorphic to $U(2)^3$.

We thus see that

$$q_{N \oplus E_8} = -q_{\pi_*(L)} = (q_{U(2)})^3. \quad \text{Q.E.D.}$$

6. Shioda-Inose structures

Definition 6.1. A K3 surface X *admits a Shioda-Inose structure* if there is a Nikulin involution ι on X with rational quotient map $\pi: X \dashrightarrow Y$ such that Y is a Kummer surface, and π_* induces a Hodge isometry $T_X(2) \cong T_Y$.

Remark 6.2. If X admits a Shioda-Inose structure, let Z be the complex torus whose Kummer surface is Y. This gives a diagram

$$\begin{tikzcd} X \arrow[dr, dashed] & & Z \arrow[dl, dashed] \\ & Y & \end{tikzcd}$$

of rational maps of degree 2. $T_X(2) \cong T_Y$ by definition, and $T_Z(2) \cong T_Y$ by Proposition 4.3. Thus, this diagram induces a Hodge isometry $T_X \cong T_Z$.

Theorem 6.3. *Let X be an algebraic K3 surface. Then the following are equivalent:*

(i) *X admits a Shioda-Inose structure.*
(ii) *There exists an abelian surface A and a Hodge isometry $T_X \cong T_A$.*
(iii) *There is a primitive embedding $T_X \hookrightarrow U^3$.*
(iv) *There is an embedding $E_8(-1)^2 \hookrightarrow NS(X)$.*

Proof. (i)$\Rightarrow$(ii) follows from Remark 6.2; the complex torus Z is an abelian surface because the Hodge structure $T_Z \cong T_X$ is polarized.

(ii)$\Rightarrow$(iii): If $T_X \cong T_A$, the natural primitive embedding $T_A \hookrightarrow H^2(A, \mathbf{Z}) \cong U^3$ induces a primitive embedding $T_X \hookrightarrow U^3$.

(iii)$\Rightarrow$(iv): We extend the given primitive embedding $\phi: T_X \hookrightarrow U^3$ to an embedding

$$\phi \oplus 0: T_X \hookrightarrow U^3 \oplus E_8(-1)^2 \cong \Lambda.$$

Since X is algebraic and $\rho(X) \geqq 17$, by Corollary 2.10, the lattice T_X admits a unique primitive embedding into the K3 lattice Λ. Thus, the embedding $\phi \oplus 0$ is isomorphic to the canonical embedding; in particular,

$$E_8(-1)^2 \hookrightarrow T_X^{\perp} = NS(X).$$

(iv)$\Rightarrow$(i): By Theorem 5.7, since $E_8(-1)^2 \hookrightarrow NS(X)$, there is a Nikulin involution ι on X such that, if $\pi: X \dashrightarrow Y$ is the rational quotient map, then π_* induces a Hodge isometry $T_X(2) \cong T_Y$, and there is a primitive embedding $N \oplus E_8(-1) \hookrightarrow NS(Y)$. Note that Y is an algebraic K3 surface and $\rho(Y) \geqq 17$. Hence, $NS(Y)$ is uniquely determined by its signature and discriminant-form (Corollary 2.10). Furthermore, $N \oplus E_8(-1)$ and the Kummer lattice K have isomorphic discriminant-forms (by (4.3)(ii) and (5.7)(iii)). Thus, by Lemma 2.3, the primitive embedding $N \oplus E_8(-1) \hookrightarrow NS(Y)$ determines a primitive embedding $K \hookrightarrow NS(Y)$. But now by Theorem 4.2(iv), Y is a Kummer surface. Q.E.D.

We should point out that the "algebraic" hypothesis is used in an essential way (in guaranteeing the uniqueness of the lattice $NS(Y)$, given its signature and discriminant-form). In fact, the generic K3 surface with $E_8(-1)^2$ in its Néron-Severi group has a Nikulin involution of the right type, but the quotient is not Kummer; conversely, the generic Kummer surface has no double cover which has a Nikulin involution of the right type. This happens because the lattices K and $N \oplus E_8(-1)$ are *not* isomorphic, even though they have the same signatures and discriminant-forms.

Corollary 6.4. *Let X be an algebraic K3 surface.*

(i) *If $\rho(X) = 19$ or 20, then X admits a Shioda-Inose structure.*

(ii) *If $\rho(X) = 18$, then X admits a Shioda-Inose structure if and only if $T_X \cong U \oplus T'$.*

(iii) *If $\rho(X) = 17$, then X admits a Shioda-Inose structure if and only if $T_X \cong U^2 \oplus T'$.*

Proof. This follows immediately from Theorem 6.3 and Corollary 2.6. Q.E.D.

Corollary 6.4 in the case $\rho(X) = 20$ was first proved by Shioda and Inose [18], using somewhat different methods.

7. Remarks on a conjecture of Takayuki Oda

In [13], Takayuki Oda made the following

Conjecture. *Let X be an algebraic K3 surface, and suppose that either $\rho(X) = 18$, 19, or 20, or that $\rho(X) = 17$ and the discriminant of the intersection-form on $NS(X)$ is a square. Then there exists an abelian surface A and a correspondance between X and A which induces a Hodge isometry*

$$(T_X \otimes \mathbf{Q}) \xrightarrow{\sim} (T_A \otimes \mathbf{Q}).$$

Corollary 7.1. *Oda's conjecture holds whenever $\rho = 19$ or 20.*

Proof. By Corollary 6.4, X admits a Shioda-Inose structure in this case. The Shioda-Inose structure induces such an isometry which is defined over $\mathbf{Z}$. Q.E.D.

Remark 7.2. The following hypothesis must be added to Oda's conjecture: "There exists an embedding of $\mathbb{Q}$-lattices

$$(T_X \otimes \mathbb{Q}) \hookrightarrow (U^3 \otimes \mathbb{Q})."$$

Proof. Note that since

$$(T_A \otimes \mathbb{Q}) \hookrightarrow H^2(A, \mathbb{Q}) \cong (U^3 \otimes \mathbb{Q}),$$

this hypothesis must hold for any K3 surface satisfying the conjecture. However, there exist K3 surfaces with $\rho = 17$ or 18 which do not satisfy this hypothesis: if T is a lattice of signature (2, 2) which does not represent zero over $\mathbb{Q}$, then $T \otimes \mathbb{Q}$ has no such embedding, by Corollary 2.7. On the other hand, by Corollary 2.10, T is the transcendental lattice of some K3 surface with Picard number 18. (There is a similar construction for $\rho = 17$.) Q.E.D.

Remark 7.3. When $\rho = 17$, the hypothesis in Oda's conjecture that the discriminant of the intersection-form on $NS(X)$ be a square is unnecessary.

Proof. In case $\rho = 17$, X admits a Shioda-Inose structure if and only if $T_X \cong U^2 \oplus T'$, where T' is a negative even lattice of rank 1; such an X will satisfy the conclusion of Oda's conjecture. On the other hand, any positive even integer $2k$ defines a negative rank 1 even lattice $T' = \langle -2k \rangle$, and the lattice

$$T_k = U^2 \oplus \langle -2k \rangle$$

occurs as the transcendental lattice of some algebraic K3 surface X by Corollary 2.10. But now,

$$\operatorname{discr}(NS(X)) = -\operatorname{discr}(T_X) = -\operatorname{discr}(T_k) = 2k,$$

which need not be a square. Q.E.D.

We thus propose the following

Modified conjecture. *Let X be an algebraic K3 surface, and suppose that there is an embedding ϕ: $(T_X \otimes \mathbb{Q}) \hookrightarrow (U^3 \otimes \mathbb{Q})$ of $\mathbb{Q}$-lattices. Then there exists an abelian surface A and a correspondance between X and A which induces a Hodge isometry*

$$(T_X \otimes \mathbb{Q}) \xrightarrow{\sim} (T_A \otimes \mathbb{Q}).$$

Remark 7.4. The Hodge conjecture implies this "modified conjecture".

Proof. Let $T = U^3 \cap \phi(T_X \otimes \mathbb{Q})$. By Corollary 1.9(ii), there is an abelian surface A such that $T \cong T_A$. ϕ induces an isometry

$$(T_X \otimes \mathbb{Q}) \xrightarrow{\sim} (T_A \otimes \mathbb{Q})$$

which gives a class in

$$H^{2,2}(X \times A) \cap H^4(X \times A, \mathbb{Q})$$

(lying in the Künneth component $H^2(X, \mathbb{Q}) \otimes H^2(A, \mathbb{Q})$; cf. [14]). But the Hodge conjecture asserts that such a class is given by a $\mathbb{Q}$-linear combination of irreducible algebraic cycles; one of these will be a correspondance inducing the given isometry. Q.E.D.

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Added in proof

In a recent preprint entitled "On the moduli space of vector bundles on K3 surfaces and its application to the Hodge conjecture", S. Mukai has shown that if X and Y are algebraic K3 surfaces with Picard number at least 11, and if $\phi: T_X \otimes \mathbb{Q} \xrightarrow{\sim} T_Y \otimes \mathbb{Q}$ is a Hodge isometry, then there is some integer n such that $n\phi$ is induced by an algebraic cycle on $X \times Y$. The "modified conjecture" in Sect. 7 follows from this (combined with Theorem 6.3), by an argument similar to (7.4).