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Chapter VI

THE CLEMENS-SCHMID EXACT SEQUENCE AND APPLICATIONS

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Let $\mathfrak{X}^* \to \Delta^*$ be a smooth family of complex varieties over the punctured disk, and let $\mathfrak{X} \to \Delta$ be a completion to a proper family over the disk, with Kähler total space. The Clemens-Schmid exact sequence relates the topology and Hodge theory of the central fibre of such a map to that of a smooth fibre by means of the monodromy of the family $\mathfrak{X}^* \to \Delta^*$. This sequence yields rather strong restrictions on the monodromy of such a family, and much information about the cohomology and monodromy of the smooth fibre can be derived from the central fibre alone. In this exposition, we have chosen to separate the topological and Hodge theoretic aspects of the sequence, for two reasons. The first is that setting up this exact sequence requires a great deal of linear algebra; by presenting a topological version first (and postponing the discussion of mixed Hodge structures) we sidestep certain technical complications until the reader has been (we hope) sufficiently motivated to wade through them. But a second, more important, reason is that many of the applications of the Clemens-Schmid sequence depend only on this topological version (i.e. depend on the weight filtrations and not the Hodge filtrations), a fact which is often ignored in the literature. It should be pointed out that the proof (which we do not give here) does not appear to separate into topological and Hodge theoretic parts.

Our main references have been [2], [4], [8] and [11]. The citations in the text are intended as guides to the reader, and do not pretend to assign proper credit to the many who have worked in this subject.

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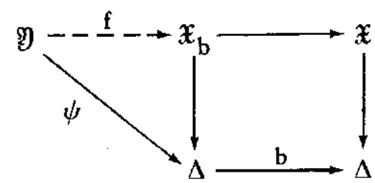
§1. *Semistable degenerations*

Let Δ denote the unit disk. A *degeneration* is a proper flat holomorphic map $\pi : \mathfrak{X} \to \Delta$ of relative dimension n such that $\mathfrak{X}_t = \pi^{-1}(t)$ is a smooth complex variety for $t \neq 0$, and $\mathfrak{X}$ is a Kähler manifold. A degeneration is *semistable* if the central fibre $\mathfrak{X}_0$ is a divisor with (global) normal crossings; in other words, writing $\mathfrak{X}_0 = \Sigma X_i$ as a sum of irreducible components, each X_i is smooth and the X_i's meet transversally so that locally π is defined by

$$t = x_1 x_2 \cdots x_k .$$

The fundamental fact about degenerations is the

SEMISTABLE REDUCTION THEOREM (Mumford [5]). *Given a degeneration* $\pi : \mathfrak{X} \to \Delta$ *there exists a base change* $b : \Delta \to \Delta$ *(defined by* $t \to t^N$ *for some* N *), a semistable degeneration* $\psi : \mathfrak{Y} \to \Delta$ *and a diagram*



such that $f : \mathfrak{Y} \dashrightarrow \mathfrak{X}_b$ *is a bimeromorphic map obtained by blowing up and blowing down subvarieties of the central fibre.*

With this theorem in hand, statements about degenerations which are invariant under blowups, blowdowns, and basechange can of course be proved by considering the special case of semistable degenerations.

Given a semistable degeneration, one can give a fairly precise description of the cohomology of its total space $\mathfrak{X}$. The first step is to construct a retraction $r : \mathfrak{X} \to \mathfrak{X}_0$ which induces isomorphisms $r^* : H^m(\mathfrak{X}_0, \mathbb{Q}) \xrightarrow{\sim} H^m(\mathfrak{X}, \mathbb{Q})$ and $r_* : H_m(\mathfrak{X}, \mathbb{Q}) \to H_m(\mathfrak{X}_0, \mathbb{Q})$. (The details of this construction can be found in Clemens [1] or Persson [8].) We then describe the cohomology of $\mathfrak{X}_0$ by means of a Mayer-Vietoris type spectral sequence, as follows.

Let

$$X_{i_0 \cdots i_p} = X_{i_0} \cap \cdots \cap X_{i_p},$$

define the *codimension* p *stratum* of $\mathfrak{X}_0$ as

$$\mathfrak{X}^{[p]} = \coprod_{i_0 < \cdots < i_p} X_{i_0 \cdots i_p}$$

(disjoint union), and let $\iota_p : \mathfrak{X}^{[p]} \to \mathfrak{X}$ be the natural map. Choose an open cover $\mathfrak{U}$ of a neighborhood of $\mathfrak{X}_0$ in $\mathfrak{X}$ such that

(1) for each $U \in \mathfrak{U}$, $\pi|_U$ is defined by

$$t = x_1 \cdots x_k$$

in suitable local coordinates

(2) $\check{H}^*(\mathfrak{U} \cap \mathfrak{X}_0, Q) \cong \check{H}^*(\mathfrak{X}_0, Q)$

(3) $\check{H}^*(\iota_p^{-1}(\mathfrak{U}), Q) = \check{H}^*(\mathfrak{X}^{[p]}, Q)$.

Let $E_0^{p,q} = \check{C}^q(\iota_p^{-1}(\mathfrak{U}), Q)$ and define differentials

$d : E_0^{p,q} \to E_0^{p,q+1}$ the Čech coboundary

$\delta : E_0^{p,q} \to E_0^{p-1,q}$ the combinatorial coboundary induced by

$$(*) \qquad \delta\phi(V \cap X_{j_0 \cdots j_{q+1}}) = \sum_\alpha (-1)^\alpha \phi(V \cap X_{j_0 \cdots \hat{j}_\alpha \cdots j_{q+1}}).$$

THEOREM. *The spectral sequence with* E_0 *term as above and*

$$E_1^{p,q} = \check{H}^q(\mathfrak{X}^{[p]}, Q)$$

degenerates at E_2, *and converges to* $\check{H}^*(\mathfrak{X}_0, Q)$.

Proof. Consider first the opposite spectral sequence

$${}^{op}E_0^{p,q} = E_0^{q,p}.$$

It is easy to check that the complex $0 \to \Gamma(\iota_0^{-1}V, \mathbb{Q}) \xrightarrow{\delta} \Gamma(\iota_1^{-1}V, \mathbb{Q}) \xrightarrow{\delta} \cdots$ has homology $\Gamma(V \cap \mathfrak{X}_0, \mathbb{Q})$ (in degree zero only), for all open sets V satisfying (1). Thus,

$$ {}^{\mathrm{op}}E_1^{p,q} = \begin{cases} \check{C}^p(\mathfrak{U} \cap \mathfrak{X}_0, \mathbb{Q}) & \text{for } q = 0 \\ 0 & \text{for } q \neq 0 \end{cases} $$

and

$$ {}^{\mathrm{op}}E_\infty^{p,q} = {}^{\mathrm{op}}E_2^{p,q} = \begin{cases} \check{H}^p(\mathfrak{X}_0, \mathbb{Q}) & \text{for } q = 0 \\ 0 & \text{for } q \neq 0 \end{cases} $$

which proves the convergence.

Note: $\mathcal{X} = \mathfrak{X}$

To prove the degeneration of the spectral sequence, we introduce its de Rham analogue. Let ${}^{DR}E_0^{p,q} = A^q(\mathfrak{X}^{[p]})$ be the complex C^∞ q-forms on $\mathcal{X}^{[p]}$, with differentials

$$ d : A^q(\mathcal{X}^{[p]}) \to A^{q+1}(\mathcal{X}^{[p]}) \quad \text{the exterior derivative} $$

$$ \delta : A^q(\mathcal{X}^{[p]}) \to A^q(\mathcal{X}^{[p+1]}) \quad \text{induced by } (*). $$

We have ${}^{DR}E_1^{p,q} = H_{DR}^q(\mathcal{X}^{[p]})$, and it is proved in Griffiths and Schmid [4, p. 71] that this spectral sequence degenerates at E_2; the essential point is the "principle of two types" for forms on a Kähler manifold. This in fact implies that the original $E_1^{p,q}$ degenerates at E_2, as we sketch below.

The differentials d_1 on $E_1^{p,q}$, $E_1^{p,q} \otimes \mathbb{C}$, and ${}^{DR}E_1^{p,q}$ are all defined by the combinatorial formula (*). We thus get a commutative diagram for $r \geq 1$

$$ \begin{array}{ccc} E_r^{p,q} & \xrightarrow{d_r} & E_r^{p,q+1} \\ \cap\downarrow & & \cap\downarrow \\ E_r^{p,q} \otimes \mathbb{C} & \xrightarrow{d_r} & E_r^{p,q+1} \otimes \mathbb{C} \\ \wr\wr & & \wr\wr \\ {}^{DR}E_r^{p,q} & \xrightarrow{d_r} & {}^{DR}E_r^{p,q+1} \end{array} $$

which implies the degeneration of $E_r^{p,q}$. q.e.d.

We use this spectral sequence to put a filtration, called the *weight filtration*, on $H^m(\mathfrak{X}_0,\mathbf{Q})$ (and hence on $H^m(\mathfrak{X},\mathbf{Q})$) as follows: define

$$W_k = \bigoplus_{q \le k} E_0^{*,q}$$

and let $W_k(H^m)$ be the induced filtration on cohomology. (This is *not* the usual filtration associated to a spectral sequence, but is more convenient here for technical reasons.) Notice that

$$O \subset W_0(H^m) \subset W_1(H^m) \subset \cdots \subset W_m(H^m) = H^m .$$

Thus, letting $Gr_k = W_k/W_{k-1}$ denote the graded pieces, we have $Gr_k(H^m) = E_2^{m-k,k}$, and $Gr_k(H^m) = 0$ if $k < 0$ or $k > m$.

We also put a weight filtration on homology $H_m(\mathfrak{X}_0,\mathbf{Q})$ (or $H_m(\mathfrak{X},\mathbf{Q})$) by duality:

$$W_{-k}(H_m) = \mathrm{Ann}(W_{k-1}(H^m)) = \{h \in H_m | (W_{k-1}(H^m),h) = 0\} .$$

With this definition,

$$Gr_k(H_m) \cong (Gr_{-k}(H^m))^*$$

so that $Gr_k(H_m) = 0$ if $k < -m$ or $k > 0$.

We conclude this section with an alternate description of the 0th graded piece of the weight filtration. Define the *dual graph* Γ of $\mathfrak{X}_0$ to be a simplicial complex with one vertex P_i for each component X_i of $\mathfrak{X}_0$, such that the simplex $<P_{i(0)},\cdots,P_{i(k)}>$ belongs to Γ if and only if $X_{i(0)}\cdots_{i(k)} \neq \emptyset$. Then $E_1^{p,0} = H^0(\mathcal{X}^{[p]})$, so that $H^0(\mathcal{X}^{[0]}) \xrightarrow{\delta} H^0(\mathcal{X}^{[1]}) \xrightarrow{\delta} \cdots$ is the Čech complex for Γ; thus $Gr_0(H^m) \cong E_2^{m,0} \cong H^m(|\Gamma|)$.

§2. *The monodromy weight filtration*

In this section and the next, all cohomology groups have $\mathbf{Q}$ coefficients unless otherwise specified.

Let $\pi : \mathfrak{X} \to \Delta$ be a degeneration, and $\pi^* : \mathfrak{X}^* \to \Delta^*$ be the restriction to the punctured disk. Fix a smooth fibre $\mathfrak{X}_t$. Since π^* is a C^∞ fibration, $\pi_1(\Delta^*)$ acts on the cohomology $H^m(\mathfrak{X}_t)$. The map

$$T : H^m(\mathfrak{X}_t) \to H^m(\mathfrak{X}_t)$$

induced by the canonical generator of $\pi_1(\Delta^*)$ is called the *Picard-Lefshetz transformation.* We have the

MONODROMY THEOREM (Landman [7]).

(1) T *is quasi-unipotent, with index of unipotency at most* m. *In other words, there is some* k *such that*

$$(T^k - I)^{m+1} = 0 \, .$$

(2) *If* $\pi : \mathfrak{X} \to \Delta$ *is semistable, then* T *is unipotent* $(k=1)$.

Thanks to this theorem, we may define the logarithm of T in the semistable case by the finite sum

$$N = \log T = (T-I) - \frac{1}{2}(T-I)^2 + \frac{1}{3}(T-I)^3 - \cdots .$$

N is nilpotent, and the index of unipotency of T coincides with the index of nilpotency of N; in particular, $T = I$ if and only if $N = 0$.

Associated to this nilpotent map N with $N^{n+1} = 0$ is an increasing filtration of $\mathbb{Q}$-subspaces

$$0 \subset W_0 \subset W_1 \subset \cdots \subset W_{2m} = H^m(\mathfrak{X}_t)$$

called the *monodromy weight filtration,* which is defined inductively as follows: first let $W_0 = \operatorname{Im} N^m$ and $W_{2m-1} = \operatorname{Ker} N^m$. Now fix some $\ell < m$; if

$$0 \subset W_{\ell-1} \subset W_{2m-\ell} \subset W_{2m} = H^m(\mathfrak{X}_t)$$

have already been defined in such a way that $N^{m-\ell+1}(W_{2m-\ell}) \subset W_{\ell-1}$, then we define

$$W_{\ell}/W_{\ell-1} = \mathrm{Im}(N^{m-2}|_{W_{2m-\ell}/W_{\ell-1}})$$

$$W_{2m-\ell-1}/W_{\ell-1} = \mathrm{Ker}(N^{m-\ell}|_{W_{2m-\ell}/W_{\ell-1}}) ,$$

and W_{ℓ}, $W_{2m-\ell-1}$ to be the corresponding inverse images. Notice that $W_{\ell}/W_{\ell-1} \subset W_{2m-\ell-1}/W_{\ell-1}$ so that $W_{\ell} \subset W_{2m-\ell-1}$. Clearly, $N^{m-1}(W_{2m-\ell-1}) \subset W_{\ell}$, so that the inductive hypotheses are satisfied.

We collect below the important linear algebra facts about this filtration; these follow from the existence of a representation of $S\ell(2;\mathbb{Q})$ on H extending the action of N. (For indications of proof, see Griffiths [3, p. 255].)

PROPOSITION. *Let* $K = \mathrm{Ker}\, N, \mathrm{Gr}_k(H) = W_k/W_{k-1}, \mathrm{Gr}_k(K) = (W_k \cap K)/(W_{k-1} \cap K)$

(1) $N(W_k) \subset W_{k-2}$

(2) $N(W_k) = (\mathrm{Im} N) \cap W_{k-2}$

(3) $N^k : \mathrm{Gr}_{m+k}(H) \xrightarrow{\sim} \mathrm{Gr}_{m-k}(H)$

(4) *Properties (1) and (3) uniquely determine the filtration.*

(5) *If* $k \le m$,

$$\mathrm{Gr}_k(H) \cong \bigoplus_{\alpha=0}^{[k/2]} \mathrm{Gr}_{k-2\alpha}(K)$$

(6) *If* $0 < k \le m$, $N^k : H^m(\mathfrak{X}_t) \to H^m(\mathfrak{X}_t)$ *is the zero map if and only if* $W_{m-k} = 0$; *if and only if* $W_{m-k} \cap K = 0$.

(7) $\mathrm{Gr}_m(H)/_{\mathrm{Im}(N : \mathrm{Gr}_{m+2}(H) \to \mathrm{Gr}_m(H))} \cong \mathrm{Gr}_m(K)$.

§3. *The Clemens-Schmid exact sequence*

Let $\mathfrak{X} \to \Delta$ be a semistable degeneration, $\mathfrak{X}_t$ a fixed smooth fibre, and $i : \mathfrak{X}_t \subset \mathfrak{X}$ the inclusion. We denote $H^m(\mathfrak{X}_t)$ by $H^m_{\lim}$, $H^m(\mathfrak{X}) \cong H^m(\mathfrak{X}_0)$ by H^m, and $H_m(\mathfrak{X}) \cong H_m(\mathfrak{X}_0)$ by H_m. $H^m_{\lim}$, H^m, and H_m all carry filtrations which we have called *weight filtrations*; we define a *weighted vector space* to be a $\mathbb{Q}$-vector space H together with an increasing filtration of $\mathbb{Q}$-subspaces

$$0 \subset \cdots \subset W_k(H) \subset W_{k+1}(H) \subset \cdots \subset H$$

called the *weight filtration.* A *morphism of weighted vector spaces of type* (r,r) is a linear map $\phi : H \to H'$ such that $\phi(W_k(H)) = W_{k+2r}(H') \cap \operatorname{Im} \phi$.

The Clemens-Schmid exact sequence studies the homomorphism $N : H^m_{\lim} \to H^m_{\lim}$ (which is a morphism of weighted vector spaces of type $(-1,1)$ by property (2) of the monodromy weight filtration). The first piece of the sequence is the

LOCAL INVARIANT CYCLE THEOREM. *The sequence*

$$H^m \xrightarrow{i^*} H^m_{\lim} \xrightarrow{N} H^m_{\lim}$$

is exact. In other words, all cocycles which are invariant under the monodromy action come from cocycles in $\mathfrak{X}$.

However, the Clemens-Schmid theorem says more. Let

$$p : H_{2n+2-m}(\mathfrak{X}) \to H^m(\mathfrak{X}, \partial\mathfrak{X})$$

and

$$p_t : H^m(\mathfrak{X}_t) \to H_{2n-m}(\mathfrak{X}_t)$$

be the Poincaré duality maps, and define $\alpha : H_{2n+2-m} \to H^m$ as the composite

$$H_{2n+2-m}(\mathfrak{X}) \xrightarrow{p} H^m(\mathfrak{X}, \partial\mathfrak{X}) \longrightarrow H^m(\mathfrak{X})$$

and $\beta : H^m_{\lim} \to H_{2n-m}$ as the composite

$$H^m(\mathfrak{X}_t) \xrightarrow{p_t} H_{2n-m}(\mathfrak{X}_t) \xrightarrow{i_*} H_{2n-m}(\mathfrak{X}) .$$

We can now state

CLEMENS-SCHMID I. *The maps* α, i^*, N, β *are morphisms of weighted vector spaces of types* $(n+1, n+1)$, $(0,0)$, $(-1,-1)$ *and* $(-n,-n)$ *respectively, and the sequence*

$$\longrightarrow H_{2n+2-m} \xrightarrow{\alpha} H^m \xrightarrow{i^*} H^m_{\lim} \xrightarrow{N} H^m_{\lim} \xrightarrow{\beta} H_{2n-m} \xrightarrow{\alpha} H^{m+2} \longrightarrow$$

is exact.

Notice that our definition of morphism included strictness (with a shift of indices) of the weight filtrations; thus, this exact sequence induces exact sequences of the filtered and graded pieces.

COROLLARY 1. *Let* K^m *denote* $\mathrm{Ker}(N : H^m_{\lim} \to H^m_{\lim})$. *Then* $W_k(H^m) \xrightarrow{\sim} W_k(K^m)$ *for* $k < m$. *In particular,* $W_{m-1}(H^m) \xrightarrow{\sim} W_{m-1}(K^m)$ *as weighted vector spaces.*

Proof. We only need to check that $W_{k-2n-2}(H_{2n+2-m}) = 0$ for $k < m$, or equivalently, that $Gr_{k-2n-2}(H_{2n+2-m}) = 0$ for all $k < m$. But in Section 1 we saw that $Gr_j(H_\ell) = 0$ if $j < -\ell$; clearly $k-2n-2 < -(2n+2-m)$. q.e.d.

Properties (3) and (5) of the monodromy weight filtration show that the graded pieces of $H^m_{\lim}$ can be recovered from the graded pieces of Ker N; the above corollary allows us to determine all but one of those pieces in terms of the weight filtration on H^m.

COROLLARY 2. *For* $k > 0$, $N^k : H^m_{\lim} \to H^m_{\lim}$ *is the zero map if and only if* $W_{m-k}(H^m) = 0$. *In particular,* N^{m+1} *is always zero, and* $N^m = 0$ *if and only if* $H^m(|\Gamma|) = 0$.

Proof. Property (6) of the monodromy weight says that $N^k = 0$ if and only if $W_{m-k}(H^m_{\lim}) = 0$. By the previous corollary, this is true if and only if $W_{m-k}(H^m) = 0$. The second statement follows from the isomorphism

$$W_0(H^m) \cong Gr_0(H^m) \cong H^m(|\Gamma|) \,. \qquad \text{q.e.d.}$$

COROLLARY 3. *The following sequence is exact*

$$0 \to Gr_{m-2}K^{m-2} \to Gr_{m-2n-2}H_{2n+2-m} \to Gr_m H^m \to Gr_m K^m \to 0 \,.$$

Proof. This follows from the strictness of morphisms in the Clemens-Schmid sequence. The only thing that requires checking is

$$\operatorname{Ker}(Gr_{m-2n-2}H_{2n+2-m} \xrightarrow{\alpha} Gr_m H^m) \cong Gr_{m-2}K^{m-2}\ .$$

But this kernel is isomorphic to

$$Gr_{m-2}H^{m-2}_{\lim} / \operatorname{Im}(Gr_m H^{m-2}_{\lim} \xrightarrow{N} Gr_{m-2}H^{m-2}_{\lim})$$

which, by property (7) of the monodromy weight filtration, is isomorphic to $Gr_{m-2}K^{m-2}$. q.e.d.

Corollary 2 allows us to compute the index of nilpotency of N from $H^m(\mathfrak{X}_0)$, and Corollary 3 (together with induction) enables us to compute the remaining graded piece of $H^m_{\lim}$. We shall apply these results in some special cases in the next section.

§4. *Applications*

(a) First cohomology groups

Since $H_{2n+1} = H_{2n+1}(\mathcal{X}_0) = 0$, the Clemens-Schmid sequence becomes

$$0 \longrightarrow H^1 \longrightarrow H^1_{\lim} \xrightarrow{N} H^1_{\lim}\ .$$

Hence, $\operatorname{Ker} N \cong H^1$ as weighted vector spaces. We compute the graded pieces

$$Gr_2 H^1_{\lim} \cong Gr_0 H^1_{\lim} \cong Gr_0 H^1 \cong H^1(|\Gamma|)$$

$$Gr_1 H^1_{\lim} \cong Gr_1 H^1 \cong \operatorname{Ker}(H^1(\mathcal{X}^{[0]}) \to H^1(\mathcal{X}^{[1]}))\ .$$

If we let $\Phi = \dim Gr_1 H^1$, then

$$b_1(\mathfrak{X}_t) = \Phi + 2h^1(|\Gamma|)$$

$$N = 0 \iff h^1(|\Gamma|) = 0\ .$$

(b) Degenerations of curves

The above analysis implies that a semistable degeneration of curves has infinite monodromy if and only if there are cycles in the dual graph of the central fibre (a classical result). The typical picture of such a degeneration is

t 0

in which the dotted cycle on $\mathfrak{X}_t$ has become a cycle in the dual graph.

In the case of curves, $\Phi = \dim H^1(\mathcal{X}^{[0]}) = 2\,\Sigma\, g(X_i)$, so that an alternate criterion for $N = 0$ is that $g(\mathfrak{X}_t) = \Sigma\, g(X_i)$.

(c) Degenerations of surfaces

Since $N = 0$ on $H^0_{\lim}$, the Clemens-Schmid sequence for H^2 breaks into two pieces:

$$0 \longrightarrow H^0 \xrightarrow{i^*} H^0_{\lim} \longrightarrow 0$$

$$0 \longrightarrow H^0_{\lim} \xrightarrow{\beta} H_4 \xrightarrow{\alpha} H^2 \xrightarrow{i^*} H^2_{\lim} \xrightarrow{N} H^2_{\lim}$$

We let

$\Phi = \dim \operatorname{Ker}(H^1(\mathcal{X}^{[0]}) \to H^1(\mathcal{X}^{[1]}))$ as above,

$q = \frac{1}{2} h^1(\mathcal{X}^{[0]})$ the sum of the irregularities of the components

$g = \frac{1}{2} h^1(\mathcal{X}^{[1]})$ the sum of the genera of the double curves.

The 0th and 1st graded pieces of $H^2_{\lim}$ can be computed by Corollary 1 of Section 3:

$$\mathrm{Gr}_0 H^2_{\lim} \cong \mathrm{Gr}_0 H^2 \cong H^2(|\Gamma|)$$

$$\mathrm{Gr}_1 H^2_{\lim} \cong \mathrm{Gr}_1 K^2 \cong \mathrm{Gr}_1 H^2 \cong H^1(\mathcal{X}^{[1]}) \Big/ \mathrm{Im}(H^1(\mathcal{X}^{[0]}) \to H^1(\mathcal{X}^{[1]})) .$$

Thus,

$$\dim \mathrm{Gr}_0 H^2_{\lim} = h^2(|\Gamma|)$$

$$\dim \mathrm{Gr}_1 H^2_{\lim} = \Phi - 2q + 2g$$

and we get the following

MONODROMY CRITERIA.

(1) $N = 0$ *on* $H^1_{\lim} \Longleftrightarrow h^1(|\Gamma|) = 0 \Longleftrightarrow b_1(\mathfrak{X}_t) = \Phi$.

(2) $N^2 = 0$ *on* $H^2_{\lim} \Longleftrightarrow h^2(|\Gamma|) = 0$.

(3) $N = 0$ *on* $H^2_{\lim} \Longleftrightarrow h^2(|\Gamma|) = 0$ *and* $\Phi + 2g = 2q$.

Computing $\mathrm{Gr}_2 H^2_{\lim}$ is somewhat more difficult. If we know the Betti numbers of the smooth fibre, the easiest way is to note that

$$b_2(\mathfrak{X}_t) = \dim \mathrm{Gr}_2 H^2_{\lim} + 2 \dim \mathrm{Gr}_1 H^2_{\lim} + 2 \dim \mathrm{Gr}_0 H^2_{\lim}$$

so that

$$\dim \mathrm{Gr}_2 H^2_{\lim} = b_2(\mathfrak{X}_t) - 2\Phi + 4q - 4g - 2h^2(|\Gamma|) .$$

To compute it directly (which will yield an expression for $b_2(\mathfrak{X}_t)$) we use

$$\mathrm{Gr}_{-4} H_4 \cong (\mathrm{Gr}_4 H^4)^* = (H^4(\mathcal{X}^{[0]}))^*$$

$$\mathrm{Gr}_2 H^2 = \mathrm{Ker}(H^2(\mathcal{X}^{[0]}) \to H^2(\mathcal{X}^{[1]}))$$

together with Corollary 3 of Section 3 to get

$$\dim \mathrm{Gr}_2 K^2 = h^0(|\Gamma|) - \#\{X_i\} + \dim \mathrm{Ker}(H^2(\mathcal{X}^{[0]}) \to H^2(\mathcal{X}^{[1]}))$$

and hence

$$\dim \mathrm{Gr}_2 H^2_{\lim} = h^2(|\Gamma|) + h^0(|\Gamma|) - \#\{X_i\} + \dim \mathrm{Ker}(H^2(\mathfrak{X}^{[0]}) \to H^2(\mathfrak{X}^{[1]})) .$$

(d) Degenerations of K3 surfaces

We will illustrate the monodromy criteria for surface degenerations with K3 surfaces. We start with the

THEOREM (Kulikov [6], Persson and Pinkham [9]). *A semistable degeneration of* K3 *surfaces is birational to one for which the central fibre* $\mathfrak{X}_0$ *is one of three types*:

Type I. $\mathfrak{X}_0$ *is a smooth* K3 *surface.*

Type II. $\mathfrak{X}_0 = X_0 \cup X_1 \cup \cdots \cup X_{k+1}$. X_α *meets only* $X_{\alpha\pm 1}$, *and each* $X_\alpha \cap X_{\alpha+1}$ *is an elliptic curve.* X_0 and X_{k+1} *are rational surfaces, and for* $1 \le \alpha \le k$, X_α *is ruled with* $X_\alpha \cap X_{\alpha-1}$ *and* $X_\alpha \cap X_{\alpha+1}$ *sections of the ruling.*

Type III. All components of $\mathfrak{X}_0$ *are rational surfaces,* $X_i \cap (\bigcup_{j\neq i} X_j)$ *is a cycle of rational curves, and* $|\Gamma| = S^2$.

We now apply the monodromy criteria in each case:

Type I. $\mathfrak{X}_0 = X$ is regular, so $q = \Phi = 0$. $\mathfrak{X}^{[1]} = \emptyset$ so $g = 0$ as well; $|\Gamma|$ is a point, so $h^2(|\Gamma|) = 0$ and we conclude $N = 0$.

Type II. $|\Gamma| = [0,1]$ so that $h^1(|\Gamma|) = h^2(|\Gamma|) = 0$. Since $b_1(\mathfrak{X}_t) = 0$, we conclude $\Phi = 0$.

The components X_0 and X_{k+1} are regular, while the X_α for $1 \le \alpha \le k$ have irregularity 1. Thus, $q = k$. The double curves all have genus 1, so that $g = k+1$. Hence, $\Phi - 2q + 2g = 2 \neq 0$ so that $N^2 = 0$ but $N \neq 0$. The monodromy weight filtration looks like:

$$0 \subset \mathbb{Q}^2 \subset \mathbb{Q}^{20} \subset \mathbb{Q}^{22} \subset \mathbb{Q}^{22} .$$

Type III. $h^2(|\Gamma|) = 1 \neq 0$ so that $N^2 \neq 0$. To compute the rest of the weight filtration, note that $\Phi = b_1(\mathfrak{X}_t) - 2h^1(|\Gamma|) = 0$; all the components are rational so that $q = 0$, and all the double curves are rational so that $g = 0$. Thus, $\dim \mathrm{Gr}_1 H^1_{\lim} = 0$ and the monodromy weight filtration looks like

$$\mathbf{Q}^1 \subset \mathbf{Q}^1 \subset \mathbf{Q}^{21} \subset \mathbf{Q}^{21} \subset \mathbf{Q}^{22} .$$

§5. *Mixed Hodge structures*

The Clemens-Schmid sequence contains more information than the topological version presented in Section 3; to explain this, we must introduce mixed Hodge structures.

A *mixed Hodge structure* is a lattice $H_{\mathbf{Z}}$ together with an increasing filtration $W_m = W_m(H)$ (called the *weight filtration*) of $H_{\mathbf{Q}} = H_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{Q}$, and a decreasing filtration $F^p = F^p(H)$ (called the *Hodge filtration*) of $H = H_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{C}$, such that the induced Hodge filtrations on $\mathrm{Gr}_m = W_m/W_{m-1}$ define a Hodge structure of weight m. More precisely, if we define

$$F^p(\mathrm{Gr}_m) = W_m \cap F^p / W_{m-1} \cap F^p$$

then

$$\mathrm{Gr}_m \cong F^p(\mathrm{Gr}_m) \oplus \overline{F^{m-p+1}(\mathrm{Gr}_m)}$$

for all p.

A *morphism of type* (r,r) between two mixed Hodge structures $H_{\mathbf{Z}}$, $W_m(H)$, $F^p(H)$ and $H'_{\mathbf{Z}}, W_m(H')$, $F^p(H')$ is a $\mathbf{Q}$-linear map

$$\phi : H_{\mathbf{Q}} \to H'_{\mathbf{Q}}$$

such that

$$\phi(W_m(H)) \subset W_{m+2r}(H')$$
$$\phi(F^p(H)) \subset F^{p+r}(H') .$$

Notice that such a morphism restricts to a morphism of type (r,r) between the weighted vector spaces $\{H_{\mathbf{Q}}, W_m(H)\}$ and $\{H'_{\mathbf{Q}}, W_m(H')\}$ (using fact (1) below for strictness).

We collect below some linear algebra facts about mixed Hodge structures: proofs can be found in [4].

PROPOSITION.

(1) *A morphism of mixed Hodge structures is strict with respect to both filtrations; in other words,*

$$\phi(W_m(H)) = W_{m+2r}(H') \cap \operatorname{Im}\phi$$

$$\phi(F^p(H)) = F^{p+r}(H') \cap \operatorname{Im}\phi .$$

(2) *If* $\phi : H_{\mathbf{Q}} \to H'_{\mathbf{Q}}$ *is a morphism of mixed Hodge structures, then the induced weight and Hodge filtrations define mixed Hodge structures on* Ker ϕ *and* Coker ϕ.

(3) *If* $H_{\mathbf{Z}}$ *carries a mixed Hodge structure, its dual* $H_{\mathbf{Z}}{}^* = \operatorname{Hom}(H_{\mathbf{Z}}, \mathbf{Z})$ *inherits a mixed Hodge structure with filtrations*

$$W_{-k}(H^*) = \operatorname{Ann}(W_{k-1}(H))$$

$$F^{-p}(H^*) = \operatorname{Ann}(F^{p+1}(H)) .$$

Each of the weighted vector spaces occurring in the Clemens-Schmid sequence actually underlies a mixed Hodge structure. As lattices, we take the integral cohomology modulo torsion; we need only define the Hodge filtrations.

For the cohomology of $\mathfrak{X}_0$, we describe the mixed Hodge structure by means of the spectral sequence

$${}^{DR}E_0^{k,\ell} = A^\ell(\mathcal{X}^{[k]}) \Rightarrow H^*(\mathcal{X}_0, C)$$

introduced in Section 1. Define

$$F^p(A^{k,\ell}) = \bigoplus_{r \geq p} H^{r,\ell-r}(\mathcal{X}^{[k]})$$

as a filtration on the E_0 term. This induces filtrations in the E_n terms and on $H^*(\mathfrak{X}_0, C)$; on the E_1 term,

$$ {}^{DR}E_1^{k,\ell} = H^{\ell}_{DR}(\mathcal{X}^{[k]}) $$

we get the usual Hodge filtration. Furthermore, the first differential d_1 gives a morphism of Hodge structures; thus,

$$ W_\ell(H^m)/_{W_{\ell-1}(H^m)} \cong {}^{DR}E_2^{m-\ell,\ell} $$

inherits a Hodge structure as well, so that $H^*(\mathfrak{X}_0,C)$ carries a mixed Hodge structure. This is a special case of a theorem of Deligne that every variety carries a canonical functorial mixed Hodge structure.

For the monodromy weight filtration, we let $\pi:\mathfrak{X}\to\Delta$ be a semistable degeneration as usual, and let $f:\mathfrak{h}\to\Delta^*$, $f(z)=e^{2\pi iz}$, be the universal cover. For each $z\in\mathfrak{h}$, there is a canonical isomorphism of $H^m(\pi^{-1}(f(z)))$ with our fixed group $H^m_{\lim}=H^m(\mathcal{X}_t)$. In particular, there are Hodge filtrations $F^p(z)$ on $H^m_{\lim}$ with the property that $TF^p(z)=F^p(z+1)$.

THEOREM (Schmid [10]). *The limit*

$$ F^p_\infty = \lim_{\mathrm{Im}(z)\to\infty} \exp(-zN)\,F^p(z) $$

exists, and the filtrations F^p_∞ *and* $W_k(H^m_{\lim})$ *define a mixed Hodge structure on* $H^m_{\lim}$, *called the* limiting mixed Hodge structure. *Furthermore,* $N:H^m_{\lim}\to H^m_{\lim}$ *is a morphism of mixed Hodge structures of type* $(-1,-1)$.

We can now state the Hodge theoretic version of the Clemens-Schmid sequence:

CLEMENS-SCHMID II. *The morphisms in the Clemens-Schmid sequence are morphisms of mixed Hodge structures (of the appropriate types).*

§6. *Further applications*

(a) Degenerations of surfaces

We represent the limiting mixed Hodge structure on $H^2_{\lim}$ pictorially as follows

$$\begin{array}{ccccccc}
 & & H^{2,2} & & & & Gr_4 \\
 & H^{2,1} & & H^{1,2} & & & Gr_3 \\
H^{2,0} & & H^{1,1} & & H^{0,2} & & Gr_2 \\
 & H^{1,0} & & H^{0,1} & & & Gr_1 \\
 & & H^{0,0} & & & & Gr_0
\end{array}$$

so that

$$F^2 \cong H^{2,0} \oplus H^{2,1} \oplus H^{2,2}$$

$$F^1/_{F^2} \cong H^{1,0} \oplus H^{1,1} \oplus H^{1,2}$$

$$F^1/_{F^1} \cong H^{0,0} \oplus H^{0,1} \oplus H^{0,2} \, .$$

Since $N : H^{2,1} \xrightarrow{\sim} H^{1,0}$ and $N^2 : H^{2,2} \xrightarrow{\sim} H^{0,0}$, we get a formula

$$h^{2,0}(\mathfrak{X}_t) = \dim F^2 Gr_2(H^2_{\lim}) + \left(\frac{1}{2}\,\Phi - q + g\right) + h^2(|\Gamma|) \, .$$

We can restrict the Clemens-Schmid sequence to

$$F^{-1}Gr_{-4}H_4 \longrightarrow F^2 Gr_2 H^2 \longrightarrow F^2 Gr_2 H^2_{\lim} \xrightarrow{N} F^1 Gr_0 H^2_{\lim} \, .$$

But

$$F^{-1}Gr_{-4}H_4 = \mathrm{Ann}(F^2 Gr_4 H^4) = 0$$

and

$$F^1 Gr_0 H^2_{\lim} = 0$$

so that

$$F^2 Gr_2 H^2_{\lim} \simeq F^2 Gr_2 H^2 \simeq H^{2,0}(\mathfrak{X}^{[0]}) \, .$$

Thus,

$$p_g(\mathfrak{X}_t) = \Sigma\, p_g(X_i) + \left(\frac{1}{2}\,\Phi - q + g\right) + h^2(|\Gamma|) \, .$$

Since $\frac{1}{2}\Phi - q + q \geq 0$, we get $p_g(\mathfrak{X}_t) \geq \Sigma\, p_g(X_i)$, and some

ALTERNATE CRITERIA FOR MONODROMY

(1) $N = 0$ on $H^1_{\lim} \Longleftrightarrow h^1(|\Gamma|) = 0$

(2) $N^2 = 0$ on $H^2_{\lim} \Longleftrightarrow h^2(|\Gamma|) = 0$

(3) $N = 0$ on $H^2_{\lim} \Longleftrightarrow p_g(\mathfrak{X}_t) = \Sigma\, p_g(X_i)$.

A formula for $h^{1,1}(\mathfrak{X}_t)$ can be derived in a similar manner, but it is usually more efficient to simply compute both $h^{2,0}(\mathfrak{X}_t)$ and $b_2(\mathfrak{X}_t)$.

(b) The geometric genus

The analysis of the geometric genus above extends easily to higher dimensions. Let $\pi : \mathfrak{X} \to \Delta$ be a semistable degeneration with fibre dimension n. As above, we have

$$F^n \cong H^{n,0}_{\lim} \oplus H^{n,1}_{\lim} \oplus \cdots \oplus H^{n,n}_{\lim} .$$

The relevant part of the Clemens-Schmid sequence is

$$F^{-1}Gr_{-n-2}H_{n+2} \longrightarrow F^nGr_nH^n \longrightarrow F_nGr_nH^n_{\lim} \xrightarrow{N} F^{n-1}Gr_{n-2}H^n_{\lim}$$

while

$$F^{n-1}Gr_{n-2}H^n_{\lim} = 0$$

and

$$F^{-1}Gr_{-n-2}H_{n+2} = \mathrm{Ann}(F^2Gr_{n+2}H^{n+2}) = 0$$

so that

$$F^nGr_nH^n_{\lim} \simeq F_nGr_nH^n = H^{n,0}(\mathfrak{X}^{[0]}) .$$

Thus, $\dim H^{n,0}_{\lim} = \Sigma\, p_g(X_i)$, so that

$$(**) \qquad p_g(\mathfrak{X}_t) \geq \Sigma\, p_g(X_i) .$$

Notice that $N = 0$ implies equality in (**). The converse is not true; however, if equality holds, then $H_{\lim}^{n,n} = H_{\lim}^{n,n-1} = 0$, which implies $H_{\lim}^{n-1,n} = 0$ and hence $Gr_n = Gr_{n-1} = 0$. Thus, we have the

GEOMETRIC GENUS CRITERION.

(1) $p_g(\mathfrak{X}_t) \geq \Sigma\, p_g(X_i)$.

(2) If $N = 0$, then equality holds in (1).

(3) If equality holds in (1), then $N^{n-1} = 0$.

REFERENCES

[1] C. H. Clemens, "Degenerations of Kähler manifolds," Duke Math J., 44 (1977), pp. 215-290.

[2] M. Cornalba and P. Griffiths, "Some transcendental aspects of algebraic geometry," Proc. Symp. Pur. Math. 29 (1974), pp. 3-110.

[3] P. Griffiths, "Periods of integrals on algebraic manifolds: Summary of main results and discussion of open problems," Bull. A.M.S. 76 (1970), pp. 228-296.

[4] P. Griffiths and W. Schmid, "Recent developments in Hodge theory: a discussion of techniques and results," in Discrete Subgroups of Lie Groups, Bombay, Oxford Univ. Press (1973), pp. 31-127.

[5] G. Kempf et al., "Toroidal Embeddings I," Lecture Notes in Math. 339 (1973).

[6] V. Kulikov, "Degenerations of K3 surfaces and Enriques surfaces," Math USSR Izvestija 11 (1977), pp. 957-989.

[7] A. Landman, "On the Picard-Lefschetz transformations," Trans A.M.S. 181 (1973), pp. 89-126.

[8] U. Persson, "Degenerations of algebraic surfaces," Mem A.M.S. 189 (1977).

[9] U. Persson and H. Pinkham, "Degenerations of surfaces with trivial canonical bundle," Ann. of Math. 113 (1981), pp. 45-66.

[10] W. Schmid, "Variation of Hodge structure: the singularities of the period mapping," Inventiones Math. 22 (1973), pp. 211-320.

[11] L. Tu, "Variation of Hodge structure and the local Torelli problem," Thesis, Harvard University (1979).

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