Math 8 - Just Some Review

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The following theorem is one that encompasses many of the "greatest hits" of Math 8. Understanding all of its pieces is a good exercise in recalling all that we've learned this quarter and excellent practice for future courses in mathematics. Since it seemed like there was some confusion in lecture and section on some of the parts, I wrote this brief note to explain in more detail what's going on.

Definition 1. *Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$ with $0 \leq k \leq n$. We define the* binomial coefficient $\binom{n}{k}$ *by*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

These are the integers[1] which appear in the well-known "Pascal's Triangle" and receive their name from the expansion of a binomial raised to the nth power:

$$(x+1)^n = \binom{n}{n}x^n + \binom{n}{n-1}x^{n-1} + \ldots + \binom{n}{1}x + \binom{n}{0}.$$

Binomial coefficients are an essential part of many areas of mathematics, but for our uses in this note we will only need two facts. The first is that $\binom{n}{k}$ counts the number of ways to choose a k-element subset of an n-element set, which is why we typically refer to this number as "n choose k" when speaking out loud. We'll come back to this fact later.

The second thing we'll need is to notice that adding two "adjacent" binomial coefficients results in another binomial coefficient.

Claim 1. *For all $n \in \mathbb{N}$ and $k \in \mathbb{N}$ with $0 < k \leq n$,*

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

Although not terribly enlightening, the proof is somewhat pleasing, so it's left as a homework problem for this week.

Now that we have a couple of facts about binomial coefficients on hand, we're ready to state the theorem.

Theorem 1. *For each $n \in \mathbb{N}$,*

$$2^n = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n-1} + \binom{n}{n}.$$

[1] Take a moment to look back at the definition - isn't it surprising that this always turns out to be an integer?

For those of you familiar with Pascal's Triangle, this theorem is the claim that taking the sum of each row gives you a power of two. There are many ways to prove this, but the way we're interested in this time involves the power set of a finite set. Here's the outline of our argument:

1. Show that if A is an n-element set, then $\mathcal{P}(A)$ has 2^n elements.
2. Find a partition $\{\mathcal{P}_0(A), \ldots, \mathcal{P}_n(A)\}$ of $\mathcal{P}(A)$ such that $\mathcal{P}_k(A)$ has $\binom{n}{k}$ elements for each $k \in \{0, 1, \ldots, n\}$.
3. Conclude that
$$|\mathcal{P}(A)| = |\mathcal{P}_0(A)| + \ldots + |\mathcal{P}_n(A)|$$
since $\mathcal{P}(A)$ is the disjoint union of the sets in the partition[2], so
$$2^n = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n-1} + \binom{n}{n}.$$

There is a lot to define and show, but the essence of the proof is in these three points. We'll begin with the first point - a fact that we've used many times but haven't had the tools (induction!) to prove until now.

Claim 2. *For all $n \in \mathbb{N}$, if A is a set with n elements, then $\mathcal{P}(A)$ is a set with 2^n elements.*

Proof. Let $Q(n)$ be the statement "for any n-element set A, the power set $\mathcal{P}(A)$ has 2^n elements". Our goal is to prove that $Q(n)$ is true for all $n \in \mathbb{N}$.

Base Case: The only 0-element set is the empty set $\emptyset$, and we know that
$$\mathcal{P}(\emptyset) = \{\emptyset\}.$$
Hence $\mathcal{P}(\emptyset)$ has $1 = 2^0$ elements, so $Q(0)$ is true.

Inductive Step: Suppose that $Q(k)$ is true for some $k \in \mathbb{N}$. We want to show that $\mathcal{P}(A)$ has 2^{k+1} elements whenever A has $k+1$ elements. To that end, let A be a set with $k+1$ elements. Fix an element $x \in A$. Then notice that the elements of $\mathcal{P}(A)$ are precisely the subsets of A, and each of these fits into exactly two categories - the subsets of A which contain x, and those which do not. That is, the power set is the disjoint union of these two sets:
$$\mathcal{P}(A) = \{S \subset A \mid x \in S\} \cup \{S \subset A \mid x \notin S\}.$$
Since the two sets on the right-hand side are disjoint, the sum of their sizes is the size of the power set. That is,
$$|\mathcal{P}(A)| = |\{S \subset A \mid x \in S\}| + |\{S \subset A \mid x \notin S\}|.$$

Now, notice that $\{S \subset A \mid x \notin S\} = \mathcal{P}(A \backslash \{x\})$, which is the power set of an k-element set and hence has 2^k elements by the induction hypothesis. Then all that remains is to count the number

[2] In this note, we're using the notation that $|X|$ is the number of elements in the set X.

of elements in $\{S \subset A \mid x \in S\}$. We can see that this also has 2^k elements since we can define a bijection

$$f : \{S \subset A \mid x \notin S\} \to \{S \subset A \mid x \in S\}$$

by sending a subset $S \subset A$ which does not contain x to the subset $S \cup \{x\} \subset A$. This function is injective: if X and Y are subsets of A which do not contain x and $f(X) = f(Y)$, then

$$X = f(X)\backslash\{x\} = f(Y)\backslash\{x\} = Y.$$

Furthermore, it is surjective since for any $X \subset A$ such that $x \in S$, we know that $f(X\backslash\{x\}) = X$.

Therefore, $\{S \subset A \mid x \in S\}$ is also a 2^k-element set, so

$$|\mathcal{P}(A)| = 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}.$$

Therefore, $Q(k+1)$ is true, so $Q(k) \implies Q(k+1)$ and thus by induction, $Q(n)$ is true for all $n \in \mathbb{N}$. $\square$

So now we know that the power set of an n-element set has 2^n elements. All that remains is to find a partition of $\mathcal{P}(A)$ into parts with sizes corresponding to binomial coefficients. Like we have done in the past, we will do this by cleverly defining an equivalence relation such that the equivalence classes form the partition we want.

Definition 2. *Let A be a finite set. Define a relation R on $\mathcal{P}(A)$ by declaring*

$$X \sim Y \longleftrightarrow |X| = |Y|.$$

That is, X is related to Y if and only if there exists a bijection $f : X \to Y$.

Claim 3. *Let A be an n-element set. Then the relation R on $\mathcal{P}(A)$ described above is an equivalence relation. Moreover, the equivalence classes of R are the sets $\mathcal{P}_k(A)$, where*

$$\mathcal{P}_k(A) = \{S \subset A \mid |S| = k\}$$

for $k \in \{0, 1, \ldots, n\}$.

Proving that R is an equivalence relation is straightforward, but a good exercise, so it is not included here. Once it's proven, though, the equivalence classes must be subsets of $\mathcal{P}(A)$ where each element (which is then a subset of A) has the same size, and we can describe this by the definition of $\mathcal{P}_k(A)$ given above.

Then since we know that the set of equivalence classes for an equivalence relation forms a partition, we can conclude that

$$\{\mathcal{P}_0(A), \ldots, \mathcal{P}_n(A)\}$$

is a partition for $\mathcal{P}(A)$. Furthermore, we know that implies that these sets are pairwise disjoint, so the size of $\mathcal{P}(A)$ is equal to the sum of the sizes of each $\mathcal{P}_k(A)$. Then we need only one more piece to this puzzle:

Claim 4. *Let A be an n-element set and let $k \in \{0, 1, \ldots, n\}$. Then*

$$|\mathcal{P}_k(A)| = \binom{n}{k}.$$

Again, I will omit the proof since it's a part of your homework. But by combining all of these claims, we can complete the proof of the theorem:

$$\begin{aligned} 2^n &= |\mathcal{P}(A)| \\ &= |\mathcal{P}_0(A)| + \ldots + |\mathcal{P}_n(A)| \\ &= \binom{n}{0} + \ldots + \binom{n}{n} \end{aligned}$$