Nonlinear dispersive equations arise as models of several physical phenomena, for instance wave propagation in media such as liquids, gases and plasmas [19]. One of the most famous dispersive models is the Korteweg-de Vries (KdV) equation

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0, \quad x, t \in \mathbb{R}, \tag{1}$$

which was derived in 1895 [13] to describe unidirectional long waves propagating in a shallow channel. The inverse scattering method was first developed in the 1960's [5] to solve the initial value problem (IVP) for the KdV equation.

The study of the qualitative properties of solutions to dispersive models has since attracted considerable attention. In the last twenty years several remarkable results have been attained concerning: local and global well-posedness under minimal regularity assumptions on the initial data, blow-up profiles and global in time behavior of solutions, the stability of special solutions, among others.

My research focuses on smoothing properties of solutions to equations of KdV type, which play an important role in the well-posedness and control theory of such equations. In what follows we make use of the classical Sobolev spaces defined via the Fourier transform as

$$H^s(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : (1 + |\xi|^2)^{s/2} \hat{f} \in L^2(\mathbb{R}) \right\} \quad \text{for } s \geq 0. \tag{2}$$

For data $v_0 \in H^s(\mathbb{R})$, the solution $v(x,t) = V(t)v_0(x)$ to the linear initial value problem (IVP)

$$\partial_t v + \partial_x^3 v = 0, \quad v(x,0) = v_0(x), \quad x, t \in \mathbb{R}, \tag{3}$$

persists with the exactly the same spatial regularity as measured in the Sobolev scale (2). In other words, the collection of operators $\{V(t) : t \in \mathbb{R}\}$ forms a unitary group acting on $H^s(\mathbb{R})$. However, if in addition v_0 is compactly supported, then $v(\cdot, t)$ is of class $C^\infty(\mathbb{R})$ for $t \neq 0$. Informally, decay of the initial data leads to smoothness of the solution measured in an appropriate sense.

Analogous results hold for the nonlinear KdV equation (for example [21]). Kato [9] considered initial data having exponential decay on the positive half-line and showed that the corresponding solution is of class $C^\infty(\mathbb{R})$ for $t > 0$. His proof uncovered *quasi-parabolic* behavior in the operator $\partial_t + \partial_x^3$ when studied in an exponentially weighted space. Kruzhkov and Faminskiĭ [14] established a connection between polynomial decay on the positive half-line and a gain in regularity for positive times by utilizing decay properties of the fundamental solution to the linear problem (3).

Also in [9], Kato deduces the following *local smoothing* effect: a "solution" u to equation (1) gains one derivative relative to the initial data u_0 in the sense that

$$\int_{-T}^{T} \int_{-R}^{R} (\partial_x u)^2(x,t)\, dxdt \leq c(R; T; \|u_0\|_2). \tag{4}$$

A modification of Kato's technique allowed Craig and Goodman [3] to duplicate the results of Kruzhkov and Faminskiĭ for a variable coefficient version of (3).

Recently, Isaza, Linares and Ponce [7] used Kato's weighted energy method to demonstrate a *propagation of regularity* effect for solutions to the KdV equation. Roughly, appropriate regularity in the initial data on the positive, or right, half-line travels to the left with infinite speed. Similar results hold for the completely integrable Benjamin-Ono equation [8].

The first portion of my dissertation extends this argument to the fifth order KdV equation

$$\partial_t u - \partial_x^5 u - 30u^2 \partial_x u + 20\partial_x u \partial_x^2 u + 10u\partial_x^3 u = 0, \quad x, t \in \mathbb{R}. \tag{5}$$

Kwon [15] introduced a corrected energy to establish local well-posedness for the IVP associated to equation (5) in $H^s(\mathbb{R})$ for $s > 5/2$. In collaboration with Segata [25], we incorporated elements of his proof into the inductive argument of [7] to obtain the following theorem.

Theorem 1. *Suppose $u_0 \in H^s(\mathbb{R})$, $s > 5/2$, and for some $l \in \mathbb{Z}^+, l \geq 3$,*

$$\|\partial_x^l u_0\|_{L^2(0,\infty)}^2 = \int_0^\infty (\partial_x^l u_0)^2(x)\, dx < \infty. \tag{6}$$

Then there exists $T > 0$ and a local solution u of the IVP associated to (5) (see [15]) such that

$$\sup_{0\leq t\leq T} \int_{\epsilon-\nu t}^\infty (\partial_x^l u)^2(x,t)\, dx \leq c \tag{7}$$

for any $\nu \geq 0, \epsilon > 0$, with $c = c(l;\nu;\epsilon;T;\|u_0\|_{H^{5/2+}};\|\partial_x^l u_0\|_{L^2(x_0,\infty)})$. In particular, for all $t \in (0,T]$, the restriction of $u(\cdot,t)$ to any interval (y,∞) belongs to $H^l(y,\infty)$.

Equations (1) and (5) are the second and third members, respectively, of a sequence of completely integrable equations known as the KdV hierarchy. In [25] we also explored the propagation of regularity within equations in the following family containing the KdV hierarchy

$$\partial_t u + \partial_x^{2j+1} u + P(u, \partial_x u, \ldots, \partial_x^{2j-1} u) = 0, \qquad x,t \in \mathbb{R}, \tag{8}$$

with $j \in \mathbb{Z}^+$ and $P : \mathbb{R}^{2j} \to \mathbb{R}$ a polynomial having no constant or linear terms. Kenig, Ponce and Vega [10] established local well-posedness for equations in this family in weighted Sobolev spaces. We extended Theorem 1 to the family (8) by utilizing decay properties of the solutions.

The second portion of my dissertation investigates quasilinear equations of the form

$$\partial_t u + a(u, \partial_x u, \partial_x^2 u)\partial_x^3 u + f(u, \partial_x u, \partial_x^2 u) = 0, \quad x,t \in \mathbb{R}, \tag{9}$$

where a and f are smooth in all variables, $\partial_{\partial_x^2 u} f \leq 0$ and $1/\kappa \leq a(\cdots) \leq \kappa$ for some $\kappa > 1$. Craig, Kappeler and Strauss [4] established local well-posedness for this family of equations in $H^s(\mathbb{R})$ with $s \geq 7$. In a joint work with Linares and Ponce [18], we demonstrated propagation of regularity for equations in this family. By adapting the use of nonlinear multipliers we obtained the following.

Theorem 2. *Suppose $u_0 \in H^s(\mathbb{R})$, $s \geq 7$, and for some $l \in \mathbb{Z}^+, l > s$,*

$$\|\partial_x^l u_0\|_{L^2(0,\infty)}^2 = \int_0^\infty (\partial_x^l u_0)^2(x)\, dx < \infty. \tag{10}$$

Then there exists $T > 0$ and a local solution u of the IVP associated to (9) (see [4]) such that

$$\sup_{0\leq t\leq T} \int_{\epsilon-\nu t}^\infty (\partial_x^l u)^2(x,t)\, dx < \infty \tag{11}$$

for any $\nu \geq 0, \epsilon > 0$.

This theorem suggests that propagation of regularity holds for equations for which Kato's local smoothing (4) can be obtained by energy methods. There is no requirement of complete integrability nor special properties of a fundamental solution to an associated linear problem. However, existing arguments do not apply to equations of Schrödinger type such as

$$i\partial_t u + \partial_x^2 u \pm |u|^\alpha u = 0, \quad x,t \in \mathbb{R},\ \alpha > 0. \tag{12}$$

Other questions stem from the work [18], which further developed solutions to the KdV equation showing that regularity does not propagate in the C^1 or C^∞ senses. These constructions would seem to adapt most easily to the fifth order KdV equation. It is not clear how to construct such solutions for the Benjamin-Ono equation, which corresponds to the case $a = 0$ in the family

$$\partial_t u + D^{1+a}\partial_x u + u\partial_x u = 0, \quad x, t \in \mathbb{R}. \tag{13}$$

This family of equations models vorticity waves in the coastal zone [26]. Through the Fourier transform we define $\widehat{D^s f} = |\xi|^s \hat{f}, s \in \mathbb{R}$. Note that $a = 1$ corresponds to the KdV equation, so that (13) defines a continuum of equations of increasing dispersive strength for a increasing. We remark that it is unknown whether or not propagation of regularity holds for the intermediate range $0 < a < 1$.

The family (13) exhibits unique behavior in weighted Sobolev spaces when compared to the KdV equation. For instance, the KdV equation preserves the Schwarz class, functions with infinite decay and regularity. However, Iorio [6] proved that solutions $u \in C([0, T] : H^2(\mathbb{R}))$ to the Benjamin-Ono equation with "too much" decay at three distinct times must vanish identically. The authors in [7] also obtained persistence of one-sided decay for solutions to the KdV equation; it remains open whether this persistence holds for equations in (13) with $0 \leq a < 1$.

Smoothing effects play a crucial role in the local well-posedness theory of nonlinear dispersive equations. For instance, Kato [9] used the local smoothing (4) to find weak, global solutions to the KdV equation corresponding to initial data in $L^2(\mathbb{R})$. A sharp version of Kato's local smoothing, proved in [11], was used by Kenig, Ponce and Vega [12] to prove local well-posedness of the IVP for the KdV equation using the contraction principle in $H^s(\mathbb{R})$. As mentioned previously, the contraction principle gives local well-posedness of the fifth order KdV (5) in integer-weighted Sobolev spaces. In fact, following [20], Pilod [22] demonstrated that local well-posedness for this equation cannot be obtained in $H^s(\mathbb{R})$ for any $s \in \mathbb{R}$ due to the presence of the term $u\partial_x^3 u$. However, the minimal, possibly fractional, weight needed in order to construct a solution via the contraction principle is currently unknown.

Smoothing effects are also important in the control theory of nonlinear dispersive equations. Consider the IVP for the forced KdV equation on a periodic domain

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = f, & x \in \mathbb{T}, t \geq 0, \\ u(x, 0) = u_0(x) \\ \partial_x^k u(0, t) = \partial_x^k u(2\pi, t) & \text{for } k = 0, 1, 2. \end{cases} \tag{14}$$

Russell and Zhang ([23], [24]) established exact controllability for this problem: given initial and terminal states u_0 and u_T, respectively, and a time $T > 0$, there exists a control function $f = f(x, t)$ localized in any interval $(a, b) \subset \mathbb{T}$ such that the solution to (14) satisfies $u(x, T) = u_T(x)$ for $x \in \mathbb{T}$. The nonlinear result uses the contraction principle along with a smoothing property of Bourgain ([1], [2]). Though the linear control theory of the KdV and Benjamin-Ono equations is similar, the nonlinear theory for the Benjamin-Ono equation ([17],[16]) has only recently been established due to the lack of an adequate smoothing effect. C. Flores and I have obtained preliminary results regarding the control theory for the equations (13) in the range $0 < a < 1$, which currently remains open.

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