Exact Control of the Linear Korteweg-de Vries Equation

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Abstract

Consider a system which evolves in time according to some physical law. Often in applications, a ‘user’ can alter a portion of the system with the intention of achieving a desired result. For example, though you are able to adjust the intensity of your stove top, the soup still obeys the heat equation.

This talk first motivates the control theory of PDEs with an example from numerical simulation. We then prove an exact controllability result for the linear Korteweg-de Vries equation. That is, how to construct a forcing function so as to guide the corresponding solution from a given initial state v_0 to a desired terminal state v_T in time $T > 0$.

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Evolution Equations

Let A be a linear differential operator (think $A = \partial_x^3$). Then a partial differential equation of the form

$$\partial_t u + Au = 0$$

can be classified as a linear evolution equation. The solution $u = u(x,t)$ often represents the state of a physical system. A common mathematical question is to determine the solution u to the above equation which matches a given initial state

$$u(x,0) = u_0(x).$$

Exact and Null Controllability

In this talk, we focus on the forced equation

$$\partial_t u + Au = F.$$

While the operator $\partial_t + A$ encapsulates the dynamical laws of the system, the function $F = F(x, t)$ represents some user input.

The Exact Control Problem Given a time $T > 0$ and two states u_0 and u_T, construct a control F such that the correpsonding solution to the above equation satisfies

$$u(x, 0) = u_0(x) \quad \text{and} \quad u(x, T) = u_T(x).$$

The Null Control Problem Simply take $u_0 = 0$ in the exact control problem.

Equivalence of Exact and Null Controllability

From the definitions, exact implies null controllability.

Conversely, suppose we are given two states u_0 and u_T and a time $T > 0$. If the system is time reversible and null controllable, then then there exist two controls and two solutions u_1, u_2 so that

$$\partial_t u_j + A u_j = F_j, \qquad (j = 1, 2)$$

and

$$\begin{aligned} u_1(x, 0) = 0, &\quad u_1(x, T) = u_T \\ u_2(x, 0) = u_0, &\quad u_2(x, T) = 0. \end{aligned}$$

Therefore, by linearity the system is exactly controllable:

$$\partial_t (u_1 + u_2) + A(u_1 + u_2) = F_1 + F_2.$$

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Weather Modelling

Consider a global weather model which runs very 12 hours, forecasting out to six days with 20km horizontal resolution. Current models simulate the full dynamical equations of motion, using approximations for sub-scale processes (like convection) or phenomenon on large time scales (e.g. air-ocean coupling).

The physics of a modern weather model is very good!

What are some options for improving performance?

- Increase resolution ($O(n^3)$ in space).
- Increase quality/quantity of observations (expensive).
- Work with what you got (data assimilation).

Data Assimilation

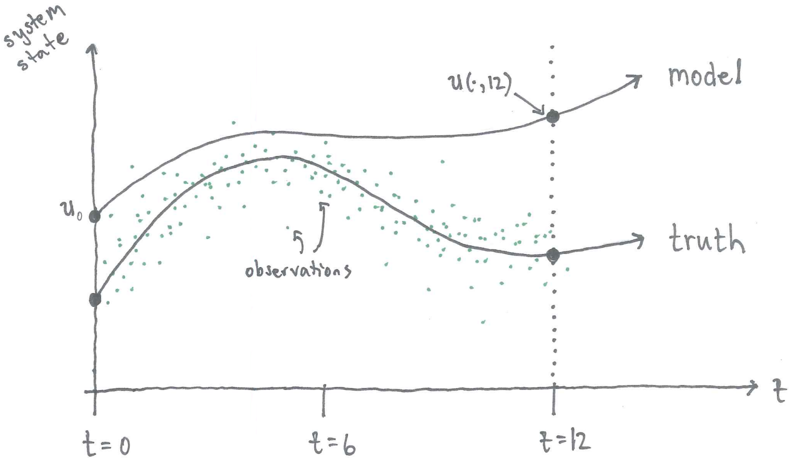

Figure : Data assimilation combines observations with previous model runs to create a more accurate initial condition for the next run.

Data Assimilation for Abstract Evolution Equations

Suppose we have observations $\tilde{u} = \tilde{u}(x, t)$ of a system over the time period $[0, T]$. To model the system for $t > T$, we must determine the state of the system at $t = T$.

The idea is choose u_0 so as to

$$\text{minimize } \|u - \tilde{u}\|,$$

where the norm $\|\cdot\|$ is unspecified and $u = u(x, t)$ solves

$$\begin{cases} \partial_t u + Au = 0 \\ u(x, 0) = u_0(x). \end{cases}$$

The dynamical laws are used as a *constraint* to determine $u(\cdot, T)$.

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The Korteweg-de Vries (KdV) Equation

The initial value problem for the KdV equation on the line is

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0, & x, t \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases}$$

This equation models the propagation of long waves in a narrow channel over a shallow bottom.

One interesting property of the KdV equation is the existence of rightward-travelling wave solutions

$$u(x,t) = \frac{3}{2} c \cdot \operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct)\right)$$

called *solitons*. Note the wave velocity $\frac{dx}{dt} = c$.

Soliton in a Narrow Channel

Figure : Recreating Scott Russell's soliton. Heriot-Watt University

The Linear KdV Equation on $\mathbb{T}$

Consider the initial value problem on a periodic domain

$$\begin{cases} \partial_t v + \partial_x^3 v = 0, \quad x \in \mathbb{T}, t \geq 0, \\ v(x,0) = v_0(x) \\ \partial_x^k v(0,t) = \partial_x^k v(2\pi, t), \quad k = 0, 1, 2. \end{cases}$$

A Special Solution

For $v_0(x) = \cos(kx)$, $k \in \mathbb{Z}$, the above problem has solution

$$v(x,t) = \cos(kx + k^3 t),$$

whose velocity $\frac{dx}{dt} = -k^2$ indicates leftward dispersion. Note that the velocity increases unbounded with the frequency. Moreover, the solution exhibits no dissipation.

Fourier Analysis of Linear KdV

Expanding the initial condition as a Fourier series

$$v_0(x) = \sum_{k \in \mathbb{Z}} \hat{v}_k e^{ikx},$$

then the function

$$v(x,t) = \sum_{k \in \mathbb{Z}} \hat{v}_k e^{i(kx + k^3 t)} =: e^{-t\partial_x^3} v_0(x)$$

solves the the linear homogeneous problem

$$\begin{cases} \partial_t v + \partial_x^3 v = 0, \quad x \in \mathbb{T}, t \geq 0, \\ v(x,0) = v_0(x) \\ \partial_x^k v(0,t) = \partial_x^k v(2\pi, t), \quad k = 0, 1, 2. \end{cases}$$

Conservation of Volume

The solution $v = v(x,t)$ to the equation $\partial_t v + \partial_x^3 v = 0$ represents the displacement of a fluid from its average height. Integrating both sides of the equation shows

$$\begin{aligned} 0 &= \int_0^{2\pi} \partial_t v(x,t)\, dx + \int_0^{2\pi} \partial_x^3 v(x,t)\, dx \\ &= \frac{d}{dt}\int_0^{2\pi} v(x,t)\, dx + \partial_x^2 v\Big|_{x=0}^{x=2\pi} \\ &= \frac{d}{dt}\int_0^{2\pi} v(x,t)\, dx. \end{aligned}$$

The volume of the fluid is constant in time.

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A Closed Loop Control

Focus on the forced equation

$$\partial_t v + \partial_x^3 v = F, \quad x \in \mathbb{T}, t \geq 0,$$

with *closed loop* control

$$F = -\kappa \left(v(x,t) - \frac{1}{2\pi} \int_0^{2\pi} v(y,t)\, dy \right), \quad \kappa > 0.$$

The quantity in parenthesis is:

- positive when $v(x,t)$ is greater than its average value;
- negative when $v(x,t)$ is less than its average value.

This control acts as a restoring force.

Volume Preservation

The function

$$F = -\kappa \left(v(x,t) - \frac{1}{2\pi} \int_0^{2\pi} v(y,t)\, dy \right), \quad \kappa > 0.$$

has average value zero.

Thus integrating both sides of the equation

$$\partial_t v + \partial_x^3 v = F, \quad x \in \mathbb{T}, t \geq 0,$$

and using tricks from prior section yields

$$\frac{d}{dt} \int_0^{2\pi} v(x,t)\, dx = 0.$$

The volume of the solution is preserved.

A Localized Closed Loop Control

We can localize this control to an interval $a \leq x \leq b$ by defining

$$g(x) = \frac{1}{b-a}\chi_{[a,b]}(x)$$

and then defining

$$F = g(x)\left(v(x,t) - \frac{1}{2\pi}\int_0^{2\pi} g(y)v(y,t)\,dy\right).$$

Since $\int_0^{2\pi} g(y)\,dy = 1$, this control still preserves the volume of the solution. Furthermore, "this sort of control can be realized, approximately, by a distributed pumping action." [2]

Physical Interpretation of Closed Loop Control

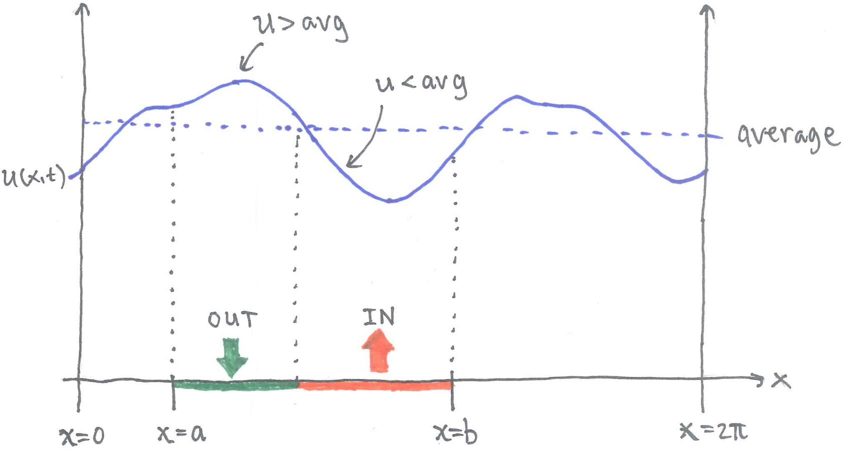

Figure : Notional depiction of distributed pumping over interval.

Stabilization

If we study the closed loop system

$$\partial_t v + \partial_x^3 v = -\kappa g(x) \left(v(x,t) - \frac{1}{2\pi} \int_0^{2\pi} g(y) v(y,t)\, dy \right),$$

then exact controllability cannot be attained.

Instead, the solution decays exponentially to a constant:

$$v(x,t) \longrightarrow \int_0^{2\pi} g(y) v_0(y)\, dy$$

as $t \to \infty$ (in an L^2-sense). This *stabilization* result is found in [3].

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Null Controllability of Linear KdV

Goal: Given any $T > 0$ and a terminal state $v_T \in L^2(0, 2\pi)$ with average value zero, construct $F = F(x, t)$ so that the solution to

$$\begin{cases} \partial_t v + \partial_x^3 v = F, \quad x \in \mathbb{T}, t \geq 0, \\ v(x, 0) = 0 \\ \partial_x^k v(0, t) = \partial_x^k v(2\pi, t), \quad k = 0, 1, 2 \end{cases}$$

satisfies

- preservation of volume $\frac{d}{dt} \int_0^{2\pi} v(x, t)\, dx = 0$;
- $v(x, T) = v_T(x)$ for all $x \in \mathbb{T}$.

This result was established by Russell and Zhang [3].

Outline of Proof of Null Controllability

1. Express v_T in terms of forcing F.
2. Restrict to volume-preserving control $F = Gh$.
3. Determine h in terms of v_T (hard part).
4. Check that construction works (omitted).

An Important Lemma

Suppose we have a solution $v = v(x,t)$ to the problem

$$\begin{cases} \partial_t v + \partial_x^3 v = F, \quad x \in \mathbb{T}, t \geq 0, \\ v(x,0) = 0 \\ \partial_x^k v(0,t) = \partial_x^k v(2\pi, t), \quad k = 0, 1, 2 \end{cases}$$

For any $T > 0$ the Fourier coefficients of $v(x, T)$

$$\hat{v}_k = \frac{1}{2\pi} \int_0^{2\pi} v(x, T) \exp(-ikx)\, dx$$

satisfy

$$\hat{v}_k = \frac{1}{2\pi} \int_0^T \exp(-ik^3(T-\tau)) \int_0^{2\pi} F(x,\tau) \exp(-ikx)\, dx d\tau.$$

This gives $v(x, T)$ in terms of F!
But how to solve for F in terms of $v(x, T)$?

A Proof of the Important Lemma

By Duhamel's forumla, the solution to the forced equation

$$\partial_t v + \partial_x^3 v = F$$

may be written as

$$v(x,t) = e^{-t\partial_x^3} v_0(x) + \int_0^t \exp(-(t-\tau)\partial_x^3) F(x,\tau)\, d\tau.$$

But $v_0 \equiv 0$. Applying the Fourier transform in the x-variable and interchanging the integrals yields the result.

An Open Loop, Volume Preserving Control

To preserve volume, we choose a forcing function of the form

$$\partial_t v + \partial_x^3 v = Gh,$$

where

$$(Gh)(x,t) := g(x)\left(h(x,t) - \int_0^{2\pi} g(y)h(y,t)\,dy\right).$$

Here, g is a fixed localization function with properties

- $g \geq 0$;
- g piecewise continuous;
- $\int_0^{2\pi} g(y)\,dy = 1$.

In effect, $h = h(x,t)$ becomes the control function to construct.

Properties of Gh

The operator G defined by

$$(Gh)(x) := g(x)\left(h(x) - \int_0^{2\pi} g(y)h(y)\,dy\right)$$

possesses the following properties:

- $G : L^2(0,2\pi) \to L^2(0,2\pi)$;
- G is linear;
- G is self-adjoint;
- G preserves volume, that is, $\int_0^{2\pi} Gh\,dx = 0$.

Choice of Control

We now make the clever guess

$$h(x,t) = \sum_{j \in \mathbb{Z}} h_j q_j(t)(G\phi_j)(x),$$

where

- the coeffcients h_j are to be determined;
- the q_j form a special basis for $L^2(0,T)$;
- $\phi_j(x) = \frac{1}{\sqrt{2\pi}} \exp(ijx)$.

Substituting this guess into our important lemma yields

$$\hat{v}_k = \frac{1}{2\pi} h_k e^{-ik^3 T} \|G\phi_k\|^2_{L^2(0,2\pi)},$$

completing the construction.

Extension to Nonlinear Problem

Russell and Zhang [4] also proved exact controllability of the nonlinear KdV equation

$$\begin{cases} \partial_t v + \partial_x^3 v + v\partial_x v = F, & x \in \mathbb{T}, t \geq 0, \\ v(x,0) = v_0 \\ \partial_x^k v(0,t) = \partial_x^k v(2\pi, t), & k = 0, 1, 2 \end{cases}$$

The approach use for the linear siutation breaks down almost immediately; exact and null controllability are no longer equivalent and Fourier analysis isn't directly applicable.

But all is not lost. A contraction mapping technique allows one to apply the linear theory to the nonlinear problem. Smoothing effects inherent in the equation are necessary to close this argument.

[1] A. K. Griffith and N. K. Nichols, *Data Assimilation Using Optimal Control Theory* (1994), available at https://www.reading.ac.uk/web/FILES/maths/10-94.pdf.

[2] D. L. Russell, *Computational study of the Korteweg-de Vries equation with localized control action*, Distributed parameter control systems (Minneapolis, MN, 1989), Lecture Notes in Pure and Appl. Math., vol. 128, Dekker, New York, 1991, pp. 195–203. MR1108858 (92c:65113)

[3] D. L. Russell and B. Y. Zhang, *Controllability and stabilizability of the third-order linear dispersion equation on a periodic domain*, SIAM J. Control Optim. **31** (1993), no. 3, 659–676, DOI 10.1137/0331030. MR1214759 (94g:93018)

[4] ______, *Exact controllability and stabilizability of the Korteweg-de Vries equation*, Trans. Amer. Math. Soc. **348** (1996), no. 9, 3643–3672, DOI 10.1090/S0002-9947-96-01672-8. MR1360229 (96m:93025)