Smoothing Effects for Linear Partial Differential Equations

Derek L. Smith
SIAM Seminar - Winter 2015
University of California, Santa Barbara

January 21, 2015

Table of Contents

The Heat Equation

Here is the *initial value problem* for the linear heat equation:

$$\begin{cases} \partial_t u = \partial_x^2 u, & x \in \mathbb{R}, t > 0, \\ u(x,0) = u_0(x) \end{cases}$$

in one spatial dimension. It is an example of an *evolution equation*. By a *local solution*, we mean a function $u = u(x,t)$ which

1. satisfies the differential relation for $(x,t) \in \mathbb{R} \times [0, T]$;
2. recovers the initial data.

Physically, $u(x,t)$ corresponds to the temperature at time t measured at position x along a thin, perfectly insulated, infinite length wire.

Local Well-Posedness of Evolution Equations

A central mathematical problem is to determine the existence of solutions to evolution equations. The initial value problem

$$\begin{cases} \partial_t u = Au, & x \in \mathbb{R}, t > 0, \\ u(x,0) = u_0(x), \end{cases}$$

is said to be *locally well-posed* (LWP) in the function space X if for every $u_0 \in X$ there exists a time $T > 0$ and a unique solution u to the equation satisfying two conditions:

1. The solution persists in the space X, that is,

$$u \in C([0,T] : X).$$

2. The solution depends continuously on the initial data u_0.

The Lebesgue Space $L^2(\mathbb{R})$

It is common to impose a finiteness condition on the initial data. The function space

$$L^2(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \;\middle|\; \int_{\mathbb{R}} |f(x)|^2 \, dx < \infty \right\}$$

captures this idea and has many nice mathematical properties.

1. It has an inner product

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x) g(x) \, dx.$$

2. The inner product defines a norm

$$\|f\|_2 = \left(\int_{\mathbb{R}} |f(x)|^2 \, dx \right)^{1/2} = \langle f, f \rangle^{1/2}.$$

The Sobolev Spaces $H^k(\mathbb{R})$

Functions in $L^2(\mathbb{R})$ may be "rough". When studying PDE, it is natural to require functions to be differentiable. This motivates the definition of the *Sobolev space*

$$H^k(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) \,\middle|\, \|f\|_2^2 + \|\partial_x f\|_2^2 + \cdots + \|\partial_x^k f\|_2^2 < \infty \right\}$$

for $k \in \mathbb{Z}^+$. The Sobolev norm is

$$\|f\|_{H^k} = \left(\|f\|_2^2 + \|\partial_x^k f\|_2^2 \right)^{1/2}.$$

Note that

$$H^{k+1} \subset H^k \subset H^{k-1} \subset \cdots \subset H^0 = L^2.$$

The Sobolev Embedding

If $f \in H^{k+1}(\mathbb{R})$, then $\partial_x^{k+1} f$ may be "rough" as it only lies in $L^2(\mathbb{R})$. The Sobolev notion of derivative is weaker than the usual limit definition.

However, $\partial_x^k f$ must be continuous and bounded, with

$$|\partial_x^k f(x)| \leq c\|f\|_{H^{k+1}}.$$

Abbreviating,

$$H^{k+1}(\mathbb{R}) \subset C_b^k(\mathbb{R}).$$

In particular, if $f \in H^1(\mathbb{R})$, then f is continuous and bounded.

The Fourier Transform on $\mathbb{R}$

Joseph Fourier employed this transform to study the heat equation. It is still actively researched in conjunction with PDE.

The spectrum of a function f is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} \, dx.$$

Often, we can recover f from its spectrum $\hat{f}$ via inversion

$$f(x) = \int_{\mathbb{R}} \hat{f}(x) e^{2\pi i x \xi} \, d\xi.$$

We review essential properties of the Fourier transform.

Properties of Fourier Transform

1. The Fourier transform is a linear isometry on $L^2(\mathbb{R})$, that is,

$$\widehat{(\alpha f + \beta g)}(\xi) = \alpha \hat{f}(\xi) + \beta \hat{g}(\xi), \qquad \alpha, \beta \in \mathbb{R},$$

   and

$$\|f\|_2 = \|\hat{f}\|_2.$$

   The above identity is the Plancherel (or Parseval) theorem.

2. Translating f by h units corresponds to multiplying its spectrum by a function of modulus one.

$$\widehat{f(x-h)}(\xi) = e^{-2\pi i h \xi} \hat{f}(\xi)$$

Properties of Fourier Transform

3. The derivative is a Fourier multiplier. Observe

$$\begin{aligned}\partial_x f(x) &= \partial_x \int_{\mathbb{R}} \hat{f}(\xi) e^{-2\pi i x \xi}\, d\xi \\ &= \int_{\mathbb{R}} \hat{f}(\xi) \partial_x e^{-2\pi i x \xi}\, d\xi \\ &= \int_{\mathbb{R}} \hat{f}(\xi)(-2\pi i \xi) e^{-2\pi i x \xi}\, d\xi.\end{aligned}$$

4. Repeating this procedure, for $k \in \mathbb{Z}^+$

$$\partial_x^k f(x) = \int_{\mathbb{R}} \hat{f}(\xi)(-2\pi i \xi)^k e^{-2\pi i x \xi}\, d\xi.$$

Sobolev Spaces via the Fourier Transform

Recall that $f \in H^k(\mathbb{R})$, $k \in \mathbb{Z}^+$, if

$$\|f\|_{H^k}^2 = \|f\|_2^2 + \|\partial_x^k f\|_2^2 < \infty.$$

Using the properties of the Fourier transform,

$$\begin{aligned} \|f\|_{H^k}^2 &= \|\hat{f}(\xi)\|_2^2 + \|(2\pi i \xi)^k \hat{f}(\xi)\|_2^2 \\ &= \int_{\mathbb{R}} |\hat{f}(\xi)|^2 + |2\pi\xi|^{2k} |\hat{f}(\xi)|^2 \, d\xi \\ &\approx \int_{\mathbb{R}} (1 + |\xi|^2)^k |\hat{f}(\xi)|^2 \, d\xi \\ &= \|(1 + |\xi|^2)^{k/2} \hat{f}(\xi)\|_2^2. \end{aligned}$$

This expression gives an alternate definition of the Sobolev norm. Intuitively, a function is k-times differentiable if its spectrum has enough decay to make these integrals finite.

Table of Contents

A Special Solution to the Heat Equation

The initial value problem

$$\begin{cases} \partial_t u = \partial_x^2 u, & x \in \mathbb{R}, t > 0, \\ u(x,0) = \cos(\lambda x) \end{cases}$$

with $\lambda > 0$ has solution

$$u(x,t) = e^{-\lambda^2 t} \cos(\lambda x).$$

The greater the frequency λ of the initial data, the greater the damping as time evolves. The heat equation is *dissipative*!

Solution of Heat Equation via Fourier Transform

Suppose $u_0 \in L^2(\mathbb{R})$. Beginning with the equation $\partial_t u = \partial_x^2 u$, apply the Fourier transform in the x-variable. Then

$$\begin{aligned}
\partial_t \hat{u}(\xi, t) &= \widehat{\partial_x^2 u}(\xi, t) \\
&= (2\pi i \xi)^2 \hat{u}(\xi, t) \\
&= -4\pi^2 \xi^2 \hat{u}(\xi, t).
\end{aligned}$$

For fixed $\xi \in \mathbb{R}$, we have the following ODE in time

$$\begin{cases}
\partial_t \hat{u}(t) = -\underbrace{4\pi^2\xi^2}_{m} \hat{u}(\xi, t), & t > 0, \\
\hat{u}(t=0) = \hat{u}_0(\xi).
\end{cases}$$

This ODE has the form $y' = -my$, which has solution

$$y(t) = c_0 e^{-mt}.$$

Solution and LWP of Heat Equation

Solving the ODE yields

$$\hat{u}(\xi, t) = \hat{u}_0(\xi)e^{-4\pi^2\xi^2 t}$$

and so by inversion

$$u(x,t) = \int_{\mathbb{R}} \hat{u}_0(\xi)\, \underbrace{e^{-4\pi^2\xi^2 t}}_{\text{Dissipation!}}\, e^{2\pi i x\xi}\, d\xi.$$

This expression provides existence and uniqueness of solutions in the function space $X = L^2(\mathbb{R})$ (or $H^k(\mathbb{R})$, $k \in \mathbb{Z}^+$). Note that

$$\|u(t)\|_2 = \|\hat{u}_0(\xi)e^{-4\pi^2\xi^2 t}\|_2, \qquad t > 0,$$

so that the solution persists in L^2 with norm decreasing in time. Similarly, the solution depends continuously on the initial data.

Smoothing Effect for the Heat Equation

A solution to the heat equation with $u_0 \in L^2(\mathbb{R})$ has an L^2-norm which decreases in time. Furthermore

$$u(\cdot, t) \in H^k(\mathbb{R}) \quad \text{for any } k \in \mathbb{Z}^+,\ t > 0.$$

The solution is smooth for $t > 0$! Why is this?

$$\begin{aligned}
\|\partial_x^k u(\cdot, t)\|_2 &= \|(2\pi i\xi)^k e^{-4\pi^2\xi^2 t}\hat{u}_0(\xi)\|_2 \\
&\le (2\pi)^k \| |\xi^k e^{-4\pi^2\xi^2 t}| \hat{u}_0(\xi)\|_2 \\
&\le c_k \|u_0\|_2.
\end{aligned}$$

An exponential function dominates any polynomial (if $t > 0$)!

To prove the smoothing effect we used the explicit solution provided by the Fourier transform. But we can arrive at the same conclusion using the PDE only.

Integration by Parts

Recall the integration by parts formula

$$\int_{\mathbb{R}} u \; dv = uv\Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} v \; du.$$

We can often assume u, v decay as $|x| \to \infty$ so that

$$\int_{\mathbb{R}} u \; dv = -\int_{\mathbb{R}} v \; du.$$

Also note, from the product rule

$$\frac{1}{2}\frac{d}{dt}(f^2) = f\frac{df}{dt}.$$

Smoothing Effect for the Heat Equation (Redux)

Suppose $u_0 \in L^2(\mathbb{R})$ and $u = u(x,t)$ is a solution to

$$\begin{cases} \partial_t u = \partial_x^2 u, & x \in \mathbb{R}, t > 0, \\ u(x,0) = u_0(x). \end{cases}$$

Multiplying the equation by u and integrating in the x-variable

$$\begin{aligned} & \int_{\mathbb{R}} u \partial_t u \, dx = \int_{\mathbb{R}} u \partial_x^2 u \, dx \\ \Rightarrow \quad & \frac{1}{2} \int_{\mathbb{R}} \partial_t (u^2) \, dx = -\int_{\mathbb{R}} \partial_x u \partial_x u \, dx \\ \Rightarrow \quad & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} u^2(x,t) \, dx = -\int_{\mathbb{R}} (\partial_x u)^2(x,t) \, dx \end{aligned}$$

In alternate notation

$$\frac{d}{dt} \|u(t)\|_2^2 = -2 \|\partial_x u(t)\|_2^2.$$

Smoothing Effect for the Heat Equation (Redux)

Integrating

$$\frac{d}{dt}\|u(t)\|_2^2 = -2\|\partial_x u(t)\|_2^2$$

in the time interval $[0, T]$, by the fundamental theorem of calculus

$$\|u(T)\|_2^2 - \|u_0\|_2^2 = -2\int_0^T \|\partial_x u(t)\|_2^2 \, dt.$$

By assumption, the left-hand side is finite, hence so is the right-hand side. This allow us to find t^* as small as desired so that

$$\|\partial_x u(t^*)\|_2^2 < \infty.$$

But now $u(\cdot, t^*) \in H^1(\mathbb{R})$, and so applying the LWP theorem again shows that the solution persists in $H^1(\mathbb{R})$ for all $t^* \leq t \leq T$.

An Iterative Argument

Suppose $u_0 \in L^2(\mathbb{R})$ and $u = u(x,t)$ is a solution to

$$\begin{cases} \partial_t u = \partial_x^2 u, & x \in \mathbb{R}, t > 0, \\ u(x,0) = u_0(x). \end{cases}$$

1. We proved that $u(\cdot, t) \in H^1(\mathbb{R})$ for any $t > 0$.
2. Differentiating the equation

   $$\partial_t(\partial_x u) = \partial_x^2(\partial_x u)$$

   shows $\partial_x u$ also solves the heat equation.
3. Which means we can apply the smoothing argument again!
4. Now $u(\cdot, t) \in H^2(\mathbb{R})$ for any $t > 0$.
5. By induction, $u(\cdot, t) \in H^k(\mathbb{R})$ for any $k \in \mathbb{Z}^+$, $t > 0$.

Heat Equation Summary

1. The initial value problem

$$\begin{cases} \partial_t u = \partial_x^2 u, & x \in \mathbb{R}, t > 0, \\ u(x,0) = u_0(x). \end{cases}$$

   is well-posed in $L^2(\mathbb{R})$ or $H^k(\mathbb{R})$.

2. The Fourier transform provides the solution

$$u(x,t) = \int_{\mathbb{R}} \hat{u}_0(\xi) e^{-4\pi^2 \xi^2 t} e^{2\pi i x \xi} \, d\xi.$$

3. Exponential decay of the Fourier multiplier shows:
   3.1 the L^2-norm decreases in time;
   3.2 the solution belongs to $H^k(\mathbb{R})$ for any $k \in \mathbb{Z}^+$, $t > 0$.
4. Using the "energy method" (a.k.a. integrating by parts) we provided an alternate proof of 3.1 and 3.2.

Table of Contents

The Airy Equation and Special Solution

Here is the initial value problem for the Airy equation:

$$\begin{cases} \partial_t u = -\partial_x^3 u, & x \in \mathbb{R}, t > 0, \\ u(x, 0) = u_0(x). \end{cases}$$

This evolution equation models waves in a narrow channel.

For example, taking initial data $u_0(x) = \cos(\lambda x)$, with $\lambda > 0$, the above initial value problem has solution

$$u(x, t) = \cos(\lambda x + \lambda^3 t).$$

Thus the wave $\cos(\lambda x)$ moves leftward with velocity $\sim \lambda^2$. As velocity depends on frequency, the Airy equation is *dispersive*!

Solution of the Airy Equation via Fourier Transform

Suppose $u_0 \in L^2(\mathbb{R})$. Beginning with the equation $\partial_t u = -\partial_x^3 u$, apply the Fourier transform in the x-variable. Then

$$\partial_t \hat{u}(\xi, t) = -(2\pi i \xi)^3 \hat{u}(\xi, t) = i(2\pi)^3 \xi^3 \hat{u}(\xi, t).$$

For fixed $\xi \in \mathbb{R}$, this is an ODE in time with solution

$$\hat{u}(\xi, t) = \hat{u}_0(\xi) e^{i(2\pi)^3 \xi^3 t}$$

and so by inversion

$$u(x, t) = \int_{\mathbb{R}} \hat{u}_0(\xi) \underbrace{e^{i(2\pi)^3 \xi^3 t}}_{\text{Dispersion!}} e^{2\pi i x \xi} \, d\xi.$$

LWP of the Airy Equation

For $u_0 \in L^2(\mathbb{R})$ or $H^k(\mathbb{R})$, $k \in \mathbb{Z}^+$, the initial value problem for the Airy equation has solution

$$u(x,t) = \int_{\mathbb{R}} \hat{u}_0(\xi) e^{i(2\pi)^3\xi^3 t} e^{2\pi i x\xi}\, d\xi.$$

This formula provides local well-posedness in these spaces with

$$\begin{aligned} \|u(t)\|_{H^k} &= \|(1+\xi^2)^{k/2} e^{i(2\pi)^3\xi^3 t}\hat{u}_0(\xi)\|_2 \\ &= \|(1+\xi^2)^{k/2}\hat{u}_0(\xi)\|_2 \\ &= \|u_0\|_{H^k}. \end{aligned}$$

That is, the solution persists in L^2 or H^k with *conserved* norm.

Conversely, if $u_0 \notin H^k$, then $u(t) \notin H^k$ for any $t \in \mathbb{R}$. *There can be no smoothing effect like that of the heat equation!*

Kato's Smoothing Argument

The persistence property for the Airy equation does not preclude all smoothing effects.

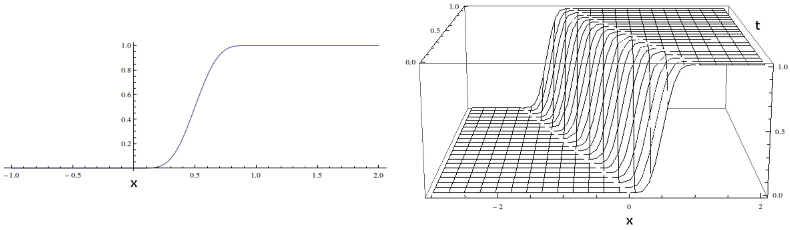

Figure : A cutoff function $\chi(x)$ and its translates $\chi(x+\nu t), \nu > 0$.

Following the intuition that high frequency waves disperse leftward more quickly than lower frequencies, Kato included a cutoff function in the energy method. The modified argument only "sees" the properties of the solution to the right.

An Example Problem

Consider the initial value problem

$$\begin{cases} \partial_t u = -\partial_x^3 u, & x \in \mathbb{R}, t > 0, \\ u(x,0) = u_0(x). \end{cases}$$

where

$$u_0(x) = \begin{cases} 1 & -1 < x < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $u_0 \in L^2(\mathbb{R})$, but it's not continuous so that $u_0 \notin H^1(\mathbb{R})$. However, u_0 is *very* smooth on the interval $[0, \infty)$.

We will show that the solution inherits this smoothness.

Smoothing Effect for the Airy Equation

Let $u = u(x,t)$ the solution to $\partial_t u = -\partial_x^3 u$ with initial data u_0. Multiplying the equation by χu and integrating in the x-variable

$$
\begin{aligned}
& \int_{\mathbb{R}} u \partial_t u \chi \, dx = - \int_{\mathbb{R}} u \partial_x^3 u \chi \, dx \\
\Rightarrow \quad & \frac{1}{2} \int_{\mathbb{R}} \partial_t (u^2 \chi) - u^2 \partial_t \chi \, dx = - \int_{\mathbb{R}} u \partial_x^3 u \chi \, dx \\
\Rightarrow \quad & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} u^2 \chi \, dx - \frac{1}{2} \int_{\mathbb{R}} u^2 \partial_t \chi \, dx = - \int_{\mathbb{R}} u \partial_x^3 u \chi \, dx \\
& \vdots
\end{aligned}
$$

After integrating by parts, we find the solution satisfies

$$
\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}} u^2 \chi(x + \nu t) \, dx + 3 \int_{\mathbb{R}} (\partial_x u)^2 \chi'(x + \nu t) \, dx \\
= \int_{\mathbb{R}} u^2 \left\{ \nu \chi'(x + \nu t) + \chi''(x + \nu t) \right\} \, dx.
\end{aligned}
$$

Smoothing Effect for the Airy Equation

Integrating in the time interval $[0, T]$, by the fundamental theorem of calculus and properties of χ,

$$\begin{aligned}\int_{\mathbb{R}} u^2(x,T)\chi(x+\nu T)\,dx &+ 3\int_0^T\int_{\mathbb{R}}(\partial_x u)^2\chi'(x+\nu t)\,dxdt \\ &\leq \int_{\mathbb{R}} u_0^2\chi(x)\,dx + c\int_0^T\int_{\mathbb{R}} u^2(x,t)\chi'(x+\nu t)\,dxdt \\ &\leq \|u_0\|_2^2 + c\int_0^T \|u(t)\|_2^2\,dt \\ &\leq (1+cT)\|u_0\|_2^2.\end{aligned}$$

Assuming $u_0 \in L^2(\mathbb{R})$, all of these expressions are finite.

Smoothing Effect for the Airy Equation

Let $u_0 \in L^2(\mathbb{R})$ and $u = u(x,t)$ be the solution to

$$\begin{cases} \partial_t u = -\partial_x^3 u, & x \in \mathbb{R}, t > 0, \\ u(x,0) = u_0(x). \end{cases}$$

Then for any $T, R > 0$

$$\int_0^T \int_{-R}^R (\partial_x u)^2(x,t)\, dxdt < \infty.$$

We gain one derivative in a *local sense*.

An Iterative Argument

Differentiating the equation, multiplying by $\partial_x u \chi$ and integrating:

$$\int_{\mathbb{R}} (\partial_x u)^2(x, T)\chi(x + \nu T)\, dx + 3 \int_0^T \int_{\mathbb{R}} (\partial_x^2 u)^2 \chi'(x + \nu t)\, dxdt$$
$$\leq \int_{\mathbb{R}} (\partial_x u_0)^2 \chi(x)\, dx + c \int_0^T \int_{-R}^R (\partial_x u)^2(x, t)\, dxdt.$$

The first term is finite by choice of u_0, the second by previous case.

Even though $u_0 \notin H^1(\mathbb{R})$, we have proved that for $x_0 \in \mathbb{R}$, $t > 0$

$$\int_{x_0}^{\infty} (\partial_x u)^2(x, t)\, dx \leq \int_{\mathbb{R}} (\partial_x u)^2(x, t)\chi(x + \nu t)\, dx < \infty.$$

Hence the restriction of $u(\cdot, t)$ to the interval (x_0, ∞) lies in $H^1(\mathbb{R})$ for $t > 0$. By induction, the restriction is smooth!

Summary

1. For $u_0 \in L^2(\mathbb{R})$, the solution u of the heat equation

$$\begin{cases} \partial_t u = \partial_x^2 u, & x \in \mathbb{R}, t > 0, \\ u(x,0) = u_0(x) \end{cases}$$

   exhibited a strong smoothing effect. For any $t > 0$, the solution $u(\cdot, t)$ lies in $H^k(\mathbb{R})$ for *any* $k \in \mathbb{Z}^+$.

2. For $u_0 \in L^2(\mathbb{R})$, the solution u of the Airy equation

$$\begin{cases} \partial_t u = -\partial_x^3 u, & x \in \mathbb{R}, t > 0, \\ u(x,0) = u_0(x). \end{cases}$$

   inherits regularity of the initial data "from the right" only:

$$u_0 \in H^k(0,\infty) \quad \Rightarrow \quad u(\cdot,t) \in H^k(x_0,\infty),\ t > 0, x_0 \in \mathbb{R}.$$

Research

Murray ([4]) analyzed the KdV equation

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0$$

with step data. Kato ([3]) proved that a solution to the KdV equation has derivatives of all orders if $e^{bm} u_0(x)$ lies in $L^2(\mathbb{R})$.

Isaza, Linares and Ponce ([1], [2]) proved versions of the theorem found in this talk for the KdV and Benjamin-Ono equations.

In an upcoming paper, Prof. Segata (Tohuku University) and myself extend these results to higher order dispersive equations like

$$\partial_t u - \partial_x^5 u + u\partial_x^3 u = 0.$$

[1] P. Isaza, F. Linares, and G. Ponce, *Propagation of regularity and decay of solutions to the k-generalized Korteweg-de Vries equation* (2014), available at http://arxiv.org/abs/1407.5110.

[2] ______, *On the propagation of regularities in solutions of the Benjamin-Ono equation* (2014), available at http://arxiv.org/abs/1409.2381.

[3] T. Kato, *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*, Studies in applied mathematics, Adv. Math. Suppl. Stud., vol. 8, Academic Press, New York, 1983, pp. 93–128. MR759907 (86f:35160)

[4] A. C. Murray, *Solutions of the Korteweg-de Vries equation from irregular data*, Duke Math. J. **45** (1978), no. 1, 149–181. MR0470533 (57 #10283)