MATH 118A, FALL 2014, PROBLEM SET 1
SOLUTIONS

1. Fun with rationals

(a) If r is rational and nonzero and x is irrational, prove that $r+x$ and rx are irrational.

Since r is rational, let it be $\frac{a}{b}$ for a, b integers with $b \neq 0$. Also $a \neq 0$ since r is nonzero.

First we show $r+x$ is irrational: We will prove this by contradiction. Suppose $r+x$ is rational. Then $r+x = \frac{m}{n}$ for some integers m, n with $n \neq 0$. Then $\frac{a}{b} + x = \frac{m}{n}$.

Rearranging, we get $x = \frac{m}{n} - \frac{a}{b} = \frac{mb-an}{nb}$. This is a ratio of integers (with the denominator not zero, since n and b aren't zero). This shows x is rational, but we are given that it is irrational. Contradiction. Thus $r+x$ must be irrational.

Next we show rx is irrational: We again prove this by contradiction. Suppose rx is rational. Then $rx = \frac{m}{n}$ for some integers m, n with $n \neq 0$. Then $\frac{ax}{b} = \frac{m}{n}$.

Rearranging: we get $x = \frac{mb}{an}$, with mb and an integers and $an \neq 0$ since $a \neq 0$ and $n \neq 0$. Thus x is rational. A contradiction again. Thus rx must be irrational.

(b) Prove that there is no rational number whose cube is 2.

We work by contradiction. Suppose there does exist $r = \frac{m}{n}$ a rational number (with m, n integers with $n \neq 0$) such that $r^3 = 2$. Since 2 is positive, r is positive, so, multiplying both by -1 if necessary, we may assume m and n are both positive.

Then we have $\frac{m^3}{n^3} = 2$. Rearranging, we have $m^3 = 2n^3$.

Now we can conclude that m^3 is even, so m is even. But then $m = 2p$ for some positive integer p (since $m \neq 0$). So $(2p)^3 = 2n^3$. So $8p^3 = 2n^3$. So $4p^3 = n^3$. Thus n is even as well. Thus $n = 2q$ for q some positive integer.

Thus $r = \frac{p}{q}$.

Consider the set S of all positive numerators of fractions whose cube is two. This is a set of positive integers, so if it is nonempty, it has a least element. But we've shown that given any fraction $\frac{m}{n}$ whose cube is two, there is one with a smaller numerator. So S does not have a least element. Thus S must be empty.

2. First notice

$$\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

This gives us the antiderivative with respect to y of the first integrand:

$$\begin{aligned}
\int_1^\infty \left(\int_1^\infty \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\right)dx &= \int_1^\infty \left(\lim_{N\to\infty}\int_1^N \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\right)dx \\
&= \int_1^\infty \lim_{N\to\infty}\left[\frac{y}{x^2+y^2}\right]_{y=1}^{y=N} dx \\
&= \int_1^\infty \left(\lim_{N\to\infty}\frac{N}{x^2+N^2}\right) - \frac{1}{x^2+1}\,dx \\
&= \int_1^\infty \frac{-1}{x^2+1}\,dx \\
&= -\lim_{N\to\infty}\int_1^N \frac{1}{x^2+1}\,dx \\
&= -\lim_{N\to\infty}\left[\arctan(x)\right]_1^N \\
&= -\lim_{N\to\infty}(\arctan(N)) + \arctan(1) \\
&= -\frac{\pi}{2} + \frac{\pi}{4} = -\frac{\pi}{4}
\end{aligned}$$

The second part is the same, except with signs reversed (since x and y have roles swapped, so the $x^2 - y^2$ gives us a different sign).

I'll do it out. First notice

$$\frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right) = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

$$\begin{aligned}
\int_1^\infty \left(\int_1^\infty \frac{x^2-y^2}{(x^2+y^2)^2}\,dx\right)dy &= \int_1^\infty \left(\lim_{N\to\infty}\int_1^N \frac{x^2-y^2}{(x^2+y^2)^2}\,dx\right)dy \\
&= \int_1^\infty \lim_{N\to\infty}\left[-\frac{x}{x^2+y^2}\right]_{x=1}^{x=N} dy \\
&= \int_1^\infty \left(\lim_{N\to\infty} - \frac{N}{N^2+x^2}\right) + \frac{1}{y^2+1}\,dy \\
&= \int_1^\infty \frac{1}{y^2+1}\,dy \\
&= \lim_{N\to\infty}\int_1^N \frac{1}{y^2+1}\,dy \\
&= \lim_{N\to\infty}\left[\arctan(y)\right]_1^N \\
&= \lim_{N\to\infty}(\arctan(N)) + \arctan(1) \\
&= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}
\end{aligned}$$

For this problem set only (i.e. not in the future), it's okay if you "plugged in" ∞. We'll talk about infinite sums and infinite integrals when we get there, though, so my doing this out properly is just a teaser.