MATH 118A, FALL 2014, PROBLEM SET 3
DUE WEDNESDAY, OCTOBER 22

1. [Rudin 2.11] Let x and y be real numbers. For each of the following functions, determine whether it satisfies the axioms of a distance function for a metric space. If it does, prove it. If not, show that it fails one of the axioms (by plugging in specific values for one of the axioms, for instance).

$$
\begin{array}{lrl}
\text{(a)} & d_1(x,y) = & |x-y|^2 \\
\text{(b)} & d_2(x,y) = & \sqrt{|x-y|} \\
\text{(c)} & d_3(x,y) = & \left|x^2-y^2\right| \\
\text{(d)} & d_4(x,y) = & |x-2y| \\
\text{(e)} & d_5(x,y) = & \dfrac{|x-y|}{1+|x-y|}
\end{array}
$$

2. Consider the following two metrics on $\mathbb{R}^2$, the euclidean distance:

$$d_{\ell^2}(x,y) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2},$$

and the taxicab distance:

$$d_{\ell^1}(x,y) = |x_1-y_1| + |x_2-y_2|\,.$$

(a) Show that $d_{\ell^2}(x,y) \leq d_{\ell^1}(x,y)$

(b) Show that $d_{\ell^1}(x,y) \leq d_{\ell^2}(x,y)\sqrt{2}$

(c) Show that the metric spaces $\mathbb{R}^2, d_{\ell^2}$ and $\mathbb{R}^2, d_{\ell^1}$ have the exact same open sets. That is, show that a set is open in one if and only if it's open in the other.

Hint: For part (c), look at the definition of open set and how it relates to open balls.

3. Determine which of the following subsets of $\mathbb{R}^2$ are open (in the usual euclidean metric on $\mathbb{R}^2$). If it's not open, name an explicit point (a,b) that shows this. If it is open, prove it.

(a) $\{(a,b) \in \mathbb{R}^2 :\ 0 < b \leq 1\}$

(b) $\mathbb{R}^2 - \{(a,b) :\ a = 0,\ -1 \leq b \leq 1\}$

(c) $\mathbb{R}^2 - \{(0,1/n) :\ n \in \mathbb{Z},\ n > 0\}$

(d) $\{(a,b) \in \mathbb{R}^2 :\ a = b\}$

(e) $\{(a,b) \in \mathbb{R}^2 :\ b < |a|\}$

Challenge (not for credit): How about the set $\mathbb{R}^2 - \{(a,b) : a > 0,\ b = \sin(1/a)\}$?