# MATH 118A, FALL 2014, HANDOUT
# SOME REAL ANALYSIS HIGHLIGHTS

1. **Intermediate value theorem:**

The intermediate value theorem states that if $f : [a, b] \to \mathbb{R}$ is a continuous real-valued function on the closed interval $[a, b]$, then the domain of $f$ contains every value between $f(a)$ and $f(b)$.

For example, if we consider $f(x) = x^2$ and the interval to be $[1, 2]$, then we have $f(1) = 1$ and $f(2) = 4$. Then we know that for every value $y \in [1, 4]$, there is some real $x \in [1, 2]$ for which $f(x) = x^2 = y$. For example, there is a real number $x$ between 1 and 2 such that $x^2 = 2$.

This is highly intuitive, but it is false if you replace the real numbers with the rational numbers. This relies on a property of the real numbers known as connectedness, and is crucial for doing any sort of work involving limits. We will develop a rigorous theory of the real numbers and we will study its "metric topology," which will lead us to rigorous proofs of theorems such as this one.

2. **Extreme value theorem:**

The simplest version of the extreme value theorem states that if $f : [a, b] \to \mathbb{R}$ is a continuous real-valued function on the closed interval $[a, b]$, then $f$ attains its maximum (and its minimum). That is, there is some value $c \in [a, b]$ such that $f(c)$ is greater than or equal to $f(x)$ for any $x \in [a, b]$.

This is a powerful theorem that relies on a concept known as compactness. Our study of metric topology and continuous functions will provide a rigorous proof of this theorem. This theorem shows up in many different contexts. If you want to find a "best" value for something, this theorem may guarantee it exists.

It also shows up in more general contexts. One famous application of compactness is to prove the *fundamental theorem of algebra*: every polynomial $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$ with complex coefficients has a complex root $r \in \mathbb{C}$ (i.e. complex number such that $f(r) = 0$). The rough outline of the proof is: there must be some $p$ such that the absolute value of $f(p)$ is smallest (this is the use of compactness, but it requires a more general version of the extreme value theorem). It turns out, if this smallest value is not zero, you can show there's some $q$ such that $f(q)$ has yet smaller absolute value, a contradiction, showing that the smallest value of the absolute value of $f(z)$ must be zero, so there is in fact a root. We will study this proof in detail later in the year.

3. **Power series:**

Our study of the convergence of infinite sums and of continuity will meet in the study of power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$, which are infinite sums of functions! This quarter, we will develop techniques to show that, when they converge, power series are continuous. Next quarter we will be able to show that in fact they are differentiable arbitrarily many times (and the differentiation is given by differentiation term by term) whenever they converge.

This will allow us to define functions like $Exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ and then rigorously prove that this is in fact the same as the function $e^x$. Similarly for the power series for $\sin x$ and $\cos x$.

4. **Swapping orders of limits:**

Swapping orders of limits or infinite sums or indefinite integrals or differentiation under the integral sign are all important tools in analysis, but they must be used with care. On your first problem set, you have an example in which swapping orders of integration is not allowed, and in fact changes the answer from negative to positive.

Here are two valid applications of swapping limits. The first is a very famous example, computing the total area under the "bell curve" and is important in probability and statistics. The second is more just for fun, packing multiple swaps in a short derivation. Each is valid, but I have omitted the justifications. We will be familiar with conditions for convergence and swapping of limits and be able to justify these by the end of the course.

Gaussian integral: $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$

$$\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 &= \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy \\
&= \int_{0}^{\infty}\int_{0}^{2\pi} e^{-r^2} r\,d\theta\,dr \qquad \text{(polar coordinates!)} \\
&= 2\pi \int_{0}^{\infty} re^{-r^2} dr = 2\pi \lim_{N\to\infty} \left[\frac{-1}{2}e^{-r^2}\right]_0^N \\
&= 2\pi \left[(\lim_{N\to\infty} \frac{-1}{2}e^{-N^2}) - \frac{-1}{2}\right] \\
&= 2\pi \left[0 + \frac{1}{2}\right] = \pi
\end{aligned}$$

Here is a calculation of $\int_0^\infty \frac{\sin x}{x}\,dx = \pi/2$ from *Mathematical Gazette* in 1909 (see `http://www.math.harvard.edu/~ctm/home/text/class/harvard/55b/10/html/home/hardy/sinx/sinx.pdf`)

$$\begin{aligned}
\int_0^\infty \frac{\sin x}{x}\,dx &= \int_0^\infty \lim_{a\to 0}\left(e^{-ax}\frac{\sin x}{x}\right) dx = \lim_{a\to 0}\int_0^\infty e^{-ax}\frac{\sin x}{x}\,dx \\
&= \lim_{a\to 0}\int_0^\infty e^{-ax}\int_0^1 \cos tx\,dt\,dx = \lim_{a\to 0}\int_0^1\int_0^\infty e^{-ax}\cos tx\,dx \\
&= \lim_{a\to 0}\int_0^1 \frac{a}{a^2+t^2}\,dt \qquad \text{(that was by parts!)} \\
&= \lim_{a\to 0}\arctan(1/a) = \pi/2
\end{aligned}$$

You need not pay attention to the derivation itself, but appreciate all the swapping of limits and indefinite integrals. This is actually a correct derivation, but without details proving that swapping the limits is allowed (which is in fact justifiable in this case).