Math 201A, Selected Homework Solutions

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Problem. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be integrable. For $\alpha \geq 0$ define $E_\alpha = \{x : |f(x)| > \alpha\}$. Prove that*

$$\int_{\mathbb{R}^d} |f| = \int_0^\infty m(E_\alpha)\, d\alpha.$$

Proof. Consider the characteristic function χ_{E_α}, which takes the value 1 when $x \in E_\alpha$ and 0 otherwise. This function is nonnegative on a σ-finite space, so by Tonelli's theorem,

$$\int_0^\infty \int_{\mathbb{R}} \chi_{E_\alpha}(x)\, dx\, d\alpha = \int_{\mathbb{R}} \int_0^\infty \chi_{E_\alpha}(x)\, d\alpha\, dx. \tag{1}$$

Notice that for fixed x, $\chi_{E_\alpha}(x)$ is 1 for $0 \leq \alpha < |f(x)|$ and 0 otherwise. Thus

$$\int_0^\infty \chi_{E_\alpha}(x)\, d\alpha = \int_0^\infty \chi_{[0,|f(x)|)}(\alpha)\, d\alpha = |f(x)|.$$

Insert this into equation (1) and note that $\int_{\mathbb{R}} \chi_{E_\alpha}(x)\, dx = m(E_\alpha)$ to obtain the result. Alternatively, since $\int_{\mathbb{R}\times[0,\infty)} |\chi_{E_\alpha}| = \int_{\mathbb{R}} |f| < \infty$, Fubini can be used instead of Tonelli. □

Problem. *Let $F \subseteq \mathbb{R}$ be closed and assume that the complement F^c has finite measure. Define functions*

$$\delta(x) = \operatorname{dist}(x, F) = \inf_{z\in F} |x - z|$$
$$I(x) = \int_{\mathbb{R}} \frac{\delta(y)}{|x-y|^2}\, dy.$$

Show that

1. $|\delta(x) - \delta(y)| \leq |x - y|$ *for all* $x, y \in \mathbb{R}$.
2. $I(x) = \infty$ *for all* $x \in F^c$.
3. $I(x) < \infty$ *for a.e.* $x \in F$.

Proof.

1. Let $x, y \in \mathbb{R}$ and $z \in F$. Notice that $\delta(x) \leq |x - z|$ by the definition of δ. From the triangle inequality we have
$$\delta(x) \leq |x - z| \leq |x - y| + |y - z|.$$
Taking the infimum of $|y - z|$ over all $z \in F$ gives $\delta(x) - \delta(y) \leq |x - y|$. Swapping the roles of x, y gives $\delta(y) - \delta(x) \leq |x - y|$, so together we find that
$$|\delta(x) - \delta(y)| \leq |x - y|.$$

2. Let $x \in F^c$. Since F is closed, F^c is open; hence we can find a ball $B_{2\epsilon}(x) \subseteq F^c$ for some $\epsilon > 0$. On the ball $B_\epsilon(x)$ we have $\delta \geq \epsilon$. Therefore,
$$I(x) = \int_{\mathbb{R}} \frac{\delta(y)}{|x-y|^2}\, dy \geq \int_{B_\epsilon(x)} \frac{\epsilon}{|x-y|^2}\, dy = 2\epsilon \int_0^\epsilon \frac{dy}{y^2} = \infty.$$

3. It suffices to show that the integral of I over F is finite. Since all functions are positive, a use of Tonelli's theorem gives

$$\int_F I(x)\,dx = \int_F \int_{\mathbb{R}} \frac{\delta(y)}{|x-y|^2}\,dy\,dx = \int_F \int_{F^c} \frac{\delta(y)}{|x-y|^2}\,dy\,dx \int_{F^c} \delta(y) \left[\int_F \frac{dx}{|x-y|^2}\right]\,dy; \tag{2}$$

the second equality comes from the fact that $\delta = 0$ on F. Given $y \in F^c$, we have $|x-y| \geq \delta(y)$ for all $x \in F$. Given such a y, let $B(y)$ denote the ball of radius $\delta(y)$ centered at y. We find that

$$\int_F \frac{dx}{|x-y|^2} \leq \int_{B(y)^c} \frac{dx}{|x-y|^2} = 2\int_{\delta(y)}^{\infty} \frac{dx}{x^2} = \frac{2}{\delta(y)}.$$

Inserting this into (2) gives

$$\int_F I(x)\,dx \leq \int_{F^c} 2\,dy = 2m(F^c) < \infty,$$

since F^c has finite measure.

□

Problem. *Let $\{K_\epsilon\}_{\epsilon>0}$ be a family of approximations to the identity on $\mathbb{R}^d$; that is, there exist constants $c_1, c_2 > 0$ so that for every $x \in \mathbb{R}^d$, $|K_\epsilon(x)| \leq c_1\epsilon^{-d}$ and $|K_\epsilon(x)| \leq c_2\epsilon|x|^{-(d+1)}$. Let $f \in L^1(\mathbb{R})$ and define the maximal function $f^*(x) = \sup_{B\ni x} m(B)^{-1}\int_B |f|$, where the supremum is over all balls in $\mathbb{R}^d$ containing the point x. Show that there exists a constant $c > 0$ so that for every $x \in \mathbb{R}^d$,*

$$\sup_{\epsilon>0} |(K_\epsilon * f)(x)| \leq cf^*(x).$$

Proof. Let $\epsilon > 0$ and $x \in \mathbb{R}^d$ be given. Write $\mathbb{R}^d = E_0 \cup (E_1 - E_0) \cup (E_2 - E_1) \cup \cdots$, where for each $k \in \mathbb{N}$, $E_k = \{y : |x-y| \leq \epsilon 2^k\}$. Denote by v_d for the volume of the unit ball in $\mathbb{R}^d$; then we have

$$\begin{aligned}
|(K_\epsilon * f)(x)| &= \left|\int K_\epsilon(x-y)f(y)\,dy\right| \\
&\leq \int_{E_0} |K_\epsilon(x-y)||f(y)|\,dy + \sum_{k=1}^{\infty}\int_{E_k - E_{k-1}} |K_\epsilon(x-y)||f(y)|\,dy \\
&\leq \frac{c_1}{\epsilon^d}\int_{E_0}|f(y)|\,dy + \sum_{k=1}^{\infty} c_2\epsilon\int_{E_k-E_{k-1}} \frac{|f(y)|}{|x-y|^{d+1}}\,dy \\
&\leq \frac{c_1 v_d}{v_d\epsilon^d}\int_{E_0}|f(y)|\,dy + \sum_{k=1}^{\infty}\frac{c_2 v_d}{v_d\epsilon^d(2^{d+1})^{k-1}}\int_{E_k-E_{k-1}}|f(y)|\,dy \\
&\leq c_1 v_d\frac{1}{m(E_0)}\int_{E_0}|f(y)|\,dy + \sum_{k=1}^{\infty}\frac{2^{-d}c_2 v_d}{2^{k-1}}\frac{1}{m(E_k)}\int_{E_k}|f(y)|\,dy \\
&\leq c_1 v_d f^*(x) + \sum_{k=1}^{\infty}\frac{2^{-d}c_2 v_d}{2^{k-1}}f^*(x) \\
&= v_d\left(c_1 + c_2 2^{-d}\right)f^*(x).
\end{aligned}$$

Take the supremum over all $\epsilon > 0$. □