1. (6 points) Answer the following questions indicating whether the statements are true or false (enclose the answer with a circle):

1.	Every nonzero square matrix has an inverse (with respect to the product of matrices):	True	False
2.	Given an $n \times n$ matrix, if $\mathbf{Ax} = 0$ only has the zero solution, then $\mathbf{Ax} = \mathbf{b}$ has a unique solution for any $\mathbf{b} \in \mathbb{R}^n$:	True	False
3.	The product of matrices is associative:	True	False
4.	The product of matrices is commutative:	True	False
5.	Given any two matrices, their product is well defined:	True	False
6.	Given any two $n \times n$ matrices, $\mathbf{A}$ and $\mathbf{B}$, $(\mathbf{AB})^T = \mathbf{A}^T\mathbf{B}^T$:	True	False

2. (10 points) Consider the following system of equations:

$$\begin{aligned} x - y - z &= 1 \\ 2x + 4y + z &= a \\ x - 4y + bz &= 3 \end{aligned}$$

1. Find all the values of a and b for which the system has a unique solution.
2. Find all the values of a and b for which the system has infinitely many solutions.
3. Find all the values of a and b for which the system does not have any solution.

Remember $A\vec{x}=\vec{B}$ has a unique solution if A^{-1} exists. Check $\det A \neq 0$ OR augment with I and RREF $[A \mid I]$.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ 1 & -4 & b & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2^* = R_2 - 2R_1} \left[\begin{array}{cccccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 6 & 3 & -2 & 1 & 0 \\ 1 & -4 & b & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3^* = R_3 - R_1} \left[\begin{array}{cccccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 6 & 3 & -2 & 1 & 0 \\ 0 & -3 & b+1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2^* = \frac{1}{6}R_2} \left[\begin{array}{cccccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{3} & 0 & 0 \\ 0 & -3 & b+1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3^* = R_3 + 3R_1]{R_1^* = R_1 + R_2} \left[\begin{array}{cccccc} 1 & 0 & -\frac{1}{2} & -\frac{2}{3} & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & b+\frac{5}{2} & -2 & 3 & 1 \end{array}\right]$$

At this point you can see that A^{-1} will not exist if $b+\frac{5}{2}=0$.

So, for a unique solution, NEED $b \neq -\frac{5}{2}$.

ANSWER: $A\vec{x}=\vec{B}$ has a unique solution for all values of a, and any value of b except $b=-\frac{5}{2}$

ANOTHER SOLUTION:

Compute $\det \begin{bmatrix} 1 & -1 & -1 \\ 2 & 4 & 1 \\ 1 & -4 & b \end{bmatrix} = 1 \begin{vmatrix} 4 & 1 \\ -4 & b \end{vmatrix} - (-1) \begin{vmatrix} 2 & 1 \\ 1 & b \end{vmatrix} - \begin{vmatrix} 2 & 4 \\ 1 & -4 \end{vmatrix}$

$$= 4b + 4 + 2b - 1 - (-8-4)$$

$$= 6b + 15$$

so $\det A \neq 0$ whenever $6b + 15 \neq 0$ i.e., $b \neq \frac{-15}{6} = \frac{-5}{2}$

Now that we know what values give us a unique (only one) solution, then if $\det A = 0$ either

(1) $A\vec{X} = \vec{B}$ has infinitely many solutions

OR (2) $A\vec{X} = \vec{B}$ has no solutions.

THIS MEANS NOW PLUG IN $b = -\frac{5}{2}$

Augment $[A|\vec{B}]$ and RREF

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 2 & 4 & 1 & a \\ 1 & -4 & -\frac{5}{2} & 3 \end{array}\right] \xrightarrow[R_3^* = R_3 - R_1]{R_2^* = R_2 - 2R_1} \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 6 & 3 & a-2 \\ 0 & -3 & -\frac{3}{2} & 2 \end{bmatrix}$$

$$\xrightarrow{R_2^* = \frac{1}{6} R_2} \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & \frac{1}{2} & \frac{a-2}{6} \\ 0 & -3 & -\frac{3}{2} & 2 \end{bmatrix}$$

$$\xrightarrow{R_3^* = R_3 + 3R_2} \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & \frac{1}{2} & \frac{a-2}{6} \\ 0 & 0 & 0 & \frac{a-2}{2} + 2 \end{bmatrix}$$

STOP! Recognize that the bottom row tells you

$$0x + 0y + 0z = \frac{a-2}{2} + 2 = \frac{a+2}{2}$$

This is ONLY TRUE for any x, y, z if $a = -2$. If $a \neq -2$, then it is NOT TRUE and impossible. So if $a \neq -2$, then $A\vec{X} = \vec{B}$ has no solution. If $a = -2$, then there are infinitely many values

REMEMBER: When RREF $[A|\vec{B}]$, if one row is all zero, then there is infinitely many solutions.

3. (9 points) Consider the following set:

$$V = \{y \in C^1(\mathbb{R}) | y(0) = 0\}.$$

Show that V is a vector space over $\mathbb{R}$. (You can assume as known the fact that $C^1(\mathbb{R})$ is a vector space)

Since $C'(\mathbb{R})$ is a vector space and V is a SUBSET of $C'(\mathbb{R})$, we can apply the theorem from class, i.e., check only the two closure properties.

(1) Closure under addition. If $f, g \in V$, is $f+g \in V$?

(i) Since $C'(\mathbb{R})$ is a vector space, then $f+g \in C'(\mathbb{R})$. ✓

(ii) Check $(f+g)(0)$.

$(f+g)(0) = f(0) + g(0) = 0 + 0 = 0$ ✓

Since (i) and (ii) are verified, then $f+g \in V$.

(2) Closure under scalar multiplication. If $a \in \mathbb{R}$, is $af \in V$?

(i) Since $C'(\mathbb{R})$ is a vector space, $af \in V$

(ii) Check $(af)(0)$.

$(af)(0) = a \cdot f(0)$

Since we have verified the two closure properties for V, V is a subspace of $C'(\mathbb{R})$. By definition, V is itself a vector space over $\mathbb{R}$.