1.
$$B = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix} \qquad A + C = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

$$AB = BA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3 \qquad |A| = 4$$

$$|AB| = 1 \qquad |BA| = 1$$

$$A^{-1} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix} \qquad A^{-1}A = AA^{-1} = I_3$$

2. 1) solution $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \text{ RREF}$$

2)

No solution

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & -4 & -3 & 0 \\ -1 & 6 & -4 & 2 \\ 1 & -1 & 0 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 4 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 40 \end{array}\right)$$

You may get different matrices here, but the conclusion is the same!

3)
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 4 \\ 3 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} \qquad \left(\begin{array}{cccc|c} 1 & 0 & 2 & -4 & 1 \\ 0 & 1 & 1 & -3 & 2 \end{array}\right)$$

where x_3, x_4 are parameters

3. I will show V is a subspace of $C'(\mathbb{R})$, and leave W is a subspace of $C'(\mathbb{R})$ as an exercise.

First of all, V is a subset of $C'(\mathbb{R})$. Assume $C'(\mathbb{R})$ is a vector space. We then only need to check the closure properties.

i) Closure under addition

For every $x(t), y(t) \in W$, we have

$x(t) \in C'(\mathbb{R})$
$y(t) \in C'(\mathbb{R})$

Since $C'(\mathbb{R})$ is a vector space, it is closed under addition, then $\boxed{x(t) + y(t) \in C'(\mathbb{R})}$

$x(1) = 0$
$y(1) = 0$

$\boxed{(x+y)(1) = x(1) + y(1) = 0 + 0 = 0}$

Then we get $x(t) + y(t) \in W$

ii) Closure under scalar multiplication

For every $y(t) \in V$, $k \in \mathbb{R}$, we have

$y(t) \in C'(\mathbb{R})$

Since $C'(\mathbb{R})$ is a vector space, it is closed under scalar multiplication, then $\boxed{ky(t) \in C'(\mathbb{R})}$

$y(1) = 0$

$\boxed{(ky)(1) = ky(1) = 0}$

Then we get $ky \in V$ #

4. In order to show S forms a basis of $\mathbb{R}^3$, we need to check the following two conditions

i) $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}$ are linearly independent

Assume $a\begin{pmatrix}1\\0\\0\end{pmatrix} + b\begin{pmatrix}1\\1\\0\end{pmatrix} + c\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

Solving this linear system, we get $a = b = c = 0$, which means they are linearly independent.

ii) $\text{span}\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\} = \mathbb{R}^3$

In order to show the above equality, we need to check

$$\mathbb{R}^3 \subseteq \text{span}\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\} \subseteq \mathbb{R}^3$$

$\boxed{\text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\} \subseteq \mathbb{R}^3}$ is trivial because

$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \in \mathbb{R}^3$. their linear combination must belong to $\mathbb{R}^3$

On the other hand, show $\mathbb{R}^3 \subseteq \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$

For every $\begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3$, we assume $\begin{pmatrix}x\\y\\z\end{pmatrix}$ can be expressed by

linear combination of $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}$, which is

$$a\begin{pmatrix}1\\0\\0\end{pmatrix} + b\begin{pmatrix}1\\1\\0\end{pmatrix} + c\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$$

From the above linear system, we get
$$a = x - y$$
$$b = y - z$$
$$c = z$$

which means every vector in $\mathbb{R}^3$ can be expressed by

the linear combination of $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}$

i.e., $\boxed{\mathbb{R}^3 \subseteq \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}}$ #

Chap 1 & 2

1. $\frac{dy}{dt} = ky(1-y)$

This is an autonomous differential equation. can be solved by seperation of variables

$$\frac{dy}{dt} = ky(1-y)$$

i) equilibrium solutions $y(t) \equiv 0$, $y(t) \equiv 1$

ii) If $y(1-y) \neq 0$, $\frac{dy}{y(1-y)} = k\,dt$ partial fractions decomption

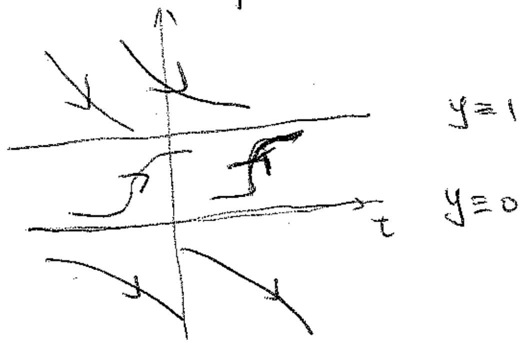
$$\frac{1}{y}dy + \frac{1}{1-y}dy = k\,dt$$

$$\ln|y| - \ln|1-y| = kt + C$$

If $0<y<1$ $\quad y(t) = \dfrac{e^{kt}\cdot e^{c}}{1+e^{kt}\cdot e^{c}}$ $\quad c \in \mathbb{R}$

If y belongs to $(1,+\infty)$ or $(-\infty,0)$, exercise !

Direction fields



$$\frac{dy}{dt} \begin{cases} - & y>1 \\ + & 0<y<1 \\ - & y<0 \end{cases}$$

stability : $y\equiv 1 \Leftrightarrow$ stable (sink)

$y\equiv 0 \Leftrightarrow$ unstable (source)

2. $y' = -\dfrac{1+y^2}{1+t^2}$ seperation of variables $\rightarrow \dfrac{dy}{1+y^2} = -\dfrac{dt}{1+t^2}$

$\rightarrow \arctan y = -\arctan t + C \quad C\in\mathbb{R}$

$\rightarrow y = \tan(-\arctan t + C) \qquad y(0) = -1 \rightarrow C = -\frac{\pi}{4}$

$\rightarrow y = \tan\left(-\arctan t - \frac{\pi}{4}\right)$

$3y^2y' - 2y^3 - t - 1 = 0$ substitution $z = y^3$

$\rightarrow z' - 2z = t+1$ Integrating Factor e^{-2t}

$\rightarrow z = -\frac{3}{4} - \frac{t}{2} + Ce^{2t} \qquad y(0) = 2 \rightarrow z(0) = 8 \rightarrow C = \frac{35}{4}$

$\rightarrow y(t) = \left(-\frac{3}{4} - \frac{t}{2} + \frac{35}{4}e^{2t}\right)^{\frac{1}{3}}$

$y' = t^2\exp(y+2t)$ seperation of variables $y(t) = -\ln\left(-\frac{1}{2}t^2e^{2t} + \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + C\right)$ $C\in\mathbb{R}$

$y' - y = \exp(3t)$ Integrating factor e^{-t} $\rightarrow y(t) = \frac{1}{2}e^{3t} + Ce^{t}$, $C\in\mathbb{R}$

$y' = 1-y^2$ seperation of variables $y\equiv\pm1$ equilibra

partial fractions decomption $-1<y<1$ $\quad y(t) = \dfrac{e^{t^2}e^{c}-1}{e^{t}e^{c}+1}$ $\quad c\in\mathbb{R}$

$y>1$ or $y<-1$ exercise

$y' = y(a - b\ln y)$ $y>0$ substitution $z = \ln y \rightarrow z' = a - bz$

Integrating factor e^{bt} $\rightarrow y(t) = e^{\frac{a}{b}}\, e^{Ce^{-bt}}$, $C\in\mathbb{R}$