§3.6 Basis and Dimension

Def: Given a vector space V and n vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$, a linear combination is a vector in V of the form

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n \qquad a_i \text{ scalars}$$

Def: We define the span of $\{v_1, v_2, \cdots, v_n\} \subseteq V$ (← subset) as the set

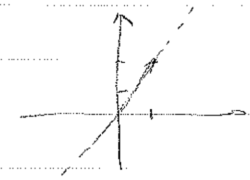
$$\text{span}\{v_1, v_2, \cdots, v_n\} = \{a_1v_1 + a_2v_2 + \cdots + a_nv_n \mid a_i \text{ scalars}\}$$

$\equiv$ set of all possible linear combinations

This is said to be generated by $\{v_1, v_2, \cdots, v_n\}$

Ex: $\mathbb{R}^2$ $\quad v_1 = \begin{pmatrix}1\\2\end{pmatrix}$

$\text{span}\{v_1\} = \left\{c\begin{pmatrix}1\\2\end{pmatrix}, c\in\mathbb{R}\right\} \equiv$ straight line



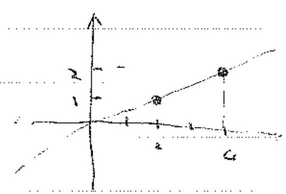
Ex: $\mathbb{R}^2$ $\quad v_1 = \begin{pmatrix}1\\0\end{pmatrix} \quad v_2 = \begin{pmatrix}0\\1\end{pmatrix}$

$\text{Span}\{v_1, v_2\} = \left\{a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix} \; a, b\in\mathbb{R}\right\} = \left\{\begin{pmatrix}a\\b\end{pmatrix} \; a, b\in\mathbb{R}\right\} = \mathbb{R}^2$

Ex: $\mathbb{R}^2$ $\quad v_1 = \begin{pmatrix}2\\1\end{pmatrix} \quad v_2 = \begin{pmatrix}4\\2\end{pmatrix}$

$\text{span}\{v_1, v_2\} = \left\{a\begin{pmatrix}2\\1\end{pmatrix} + b\begin{pmatrix}4\\2\end{pmatrix}\right\} = \left\{\begin{pmatrix}2a+4b\\a+2b\end{pmatrix}, a, b\in\mathbb{R}\right\}$

$= \left\{c\begin{pmatrix}1\\2\end{pmatrix}, c\in\mathbb{R}\right\} =$ straight line



Theorem: $\text{span}\{v_1, \cdots, v_n\} \subseteq V$ is a vector space

Proof: Given $w_1, w_2 \in \text{span}\{v_1, \cdots, v_n\}$

$w_1 = a_1v_1 + \cdots + a_nv_n$ for some $a_1, \cdots, a_n$

$w_2 = b_1v_1 + \cdots + b_nv_n$ for some $b_1, \cdots, b_n$

then $\alpha w_1 + \beta w_2 = (\alpha a_1 + \beta b_1)v_1 + \cdots + (\alpha a_n + \beta b_n)v_n$

$\in \text{span}\{v_1, v_2, \cdots, v_n\}$

Def: We say that $\{v_1, \cdots, v_n\} \subseteq V$ are linearly independent if whenever $a_1v_1 + \cdots + a_nv_n = 0$ implies $a_1 = a_2 = \cdots = a_n = 0$

Note: If $a_n \neq 0$, $v_n = -\frac{a_1}{a_n}v_1 - \cdots - \frac{a_{n-1}}{a_n}v_{n-1}$

Ex: $\mathbb{R}^3$ $\quad v_1 = \begin{pmatrix}1\\2\\3\end{pmatrix} \quad v_2 = \begin{pmatrix}4\\5\\6\end{pmatrix} \quad v_3 = \begin{pmatrix}7\\8\\9\end{pmatrix}$

Are $\vec{v_1}, \vec{v_2}$ and $\vec{v_3}$ linearly independent?

Check: Assume that there exist constants a_1, a_2, a_3 such that

$$a_1\vec{v_1} + a_2\vec{v_2} + a_3\vec{v_3} = \vec{0}$$

$$(\vec{v_1}, \vec{v_2}, \vec{v_3})\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\det\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = 0 \Rightarrow$ Infinitely many solutions, there exists a nonzero solution $a_1, a_2, a_3 \Rightarrow v_1, v_2, v_3$ are NOT linearly independent.

Ex: $v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $v_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ Are they linearly indepent?

Check: $a_1\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + a_2\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & -3 & 0 \\ 0 & -6 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

\# pivots = \# unknowns

unique solution $a_1 = 0$, $a_2 = 0$

$\Rightarrow v_1, v_2$ are linearly independent.

Ex: Consider $C(\mathbb{R})$ and $v_1(t) = e^t$ $v_2(t) = e^{-t}$

Are they linearly independent?

① Assume $a e^t + b e^{-t} = 0 \quad \forall t \in \mathbb{R}$

At $t=0$: $a + b = 0$

At $t=1$: $ae + be^{-1} = 0$

$$\Leftrightarrow \begin{pmatrix} 1 & 1 \\ e & e^{-1} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 1 \\ e & e^{-1} \end{vmatrix} = \frac{1}{e} - e \neq 0 \Rightarrow a = b = 0$$

② 2nd method. $a e^t + b e^{-t} = 0 \quad \forall t$

Taking derivative $a e^t - b e^{-t} = 0 \quad \forall t$

$$\Rightarrow \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \forall t$$

Consider $W(t) = \det\begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix} = -1 - 1 = -2 \neq 0$

$\Rightarrow a = b = 0$

Def: Given a set of functions $f_1(t), \cdots, f_n(t)$
We define the Wronskian by

$$W(t) = \begin{vmatrix} f_1(t), & \cdots & f_n(t) \\ f_1'(t), & \cdots, & f_n'(t) \\ \cdots & & \cdots \\ f_1^{(n-1)}(t), & \cdots, & f_n^{(n-1)}(t) \end{vmatrix}$$

$\{f_1, \cdots, f_n\}$ are linearly independent $\Leftrightarrow$ $W(t) \neq 0 \quad \forall t$

Ex. $f_1(x) = \cos x \quad f_2(x) = \sin x \quad f_3(x) = \cos(x-1)$

① Are f_1, f_2 linearly independent?

$$W(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0 \qquad \forall x$$

② Are $\{f_1, f_2, f_3\}$ linearly independent?

$$W(x) = \begin{vmatrix} \cos x & \sin x & \cos(x-1) \\ -\sin x & \cos x & -\sin(x-1) \\ -\cos x & -\sin x & -\cos(x-1) \end{vmatrix} = 0 \quad \forall x \Rightarrow \text{linearly dependency}$$

In fact, remember that

$$\cos(x-1) = \cos x \cos 1 + \sin x \sin 1$$
$$f_3(x) = \cos 1 \, f_1(x) + \sin 1 \, f_2(x)$$

Def: Given a vector space V, we say that a set of vectors $\{v_1, \cdots, v_n\}$ form a basis of V if
(i) $\{v_1, \cdots, v_n\}$ are linearly independent
(ii) $\mathrm{span}\{v_1, \cdots, v_n\} = V$

Def: We define the dimension of V as the number of vectors in any basis of V.
Note: This implicity assumes the fact that any basis has the same number of vectors.

Ex.: In $\mathbb{R}^2$, consider $\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
wha is $\mathrm{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

If $\vec{b} \in \mathbb{R}^2$ is in span $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$, then $\vec{b} = a_1\vec{v_1} + a_2\vec{v_2} + a_3\vec{v_3}$

$$\left(\begin{array}{ccc|c} 0 & 2 & 1 & b_1 \\ 1 & 3 & 1 & b_2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 3 & 1 & b_2 \\ 0 & 2 & 1 & b_1 \end{array}\right) \qquad \begin{array}{l} a_1 = b_2 - a_3 - 3a_2 \\ a_2 = \dfrac{b_1 - a_3}{2} \end{array}$$

$\Rightarrow$ span $\{v_1, v_2, v_3\} = \mathbb{R}^2$

ii) Are $\{v_1, v_2, v_3\}$ linearly independent?

$\vec{b} = 0 \Rightarrow a_1 v_1 + a_2 v_2 + a_3 v_3 = 0 \not\Rightarrow a_1 = a_2 = a_3 = 0$

In fact $\vec{v_3} = -\frac{1}{2}\vec{v_1} + \frac{1}{2}\vec{v_2}$ — Not linearly independent

Span $\{v_1\ v_2\ v_3\}$ = Span $\{v_1\ v_2\} = \mathbb{R}^2$

Are $\{v_1\ v_2\}$ linearly independent? $v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Consider $A = \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}$ rank$(A) = 2 \Rightarrow$ Only zero solution

$\Rightarrow v_1, v_2$ linearly independent.

$\Rightarrow \{v_1, v_2\}$ forms a basis of $\mathbb{R}^2$

Ex: Consider vector space $V = \{$polynomials of degree $\le 2\}$

$= \{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$

Generating set : $\{1, x, x^2\}$

① linearly independent?

$$W(x) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2 \neq 0 \quad \forall x$$

② span $\{1, x, x^2\} = V$ ✓

$\Rightarrow \{1, x, x^2\}$ forms a basis

Note: Generating $\mathbb{R}^n$ means being able to solve

$Ax = b \quad \forall b \in \mathbb{R}^n$ where $A = [v_1\ v_2 \cdots v_n]$

Linear independence means $Ax = 0$ only has zero solution

Ex: Consider $v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $v_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ $v_3 = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$ in $\mathbb{R}^3$

What is $V =$ span $\{v_1, v_2, v_3\}$

① $\forall b \in \mathbb{R}^3 \Rightarrow a_1 v_1 + a_2 v_2 + a_3 v_3 = b$

$$\left(\begin{array}{ccc|c} 1 & 4 & 7 & b_1 \\ 2 & 5 & 8 & b_2 \\ 3 & 6 & 9 & b_3 \end{array}\right) \xrightarrow{④} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -3b_1 + 2b_2 \\ 0 & 1 & 2 & b_1 - \frac{b_2}{2} \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{array}\right)$$

Consistency $\Rightarrow$ $b_3 - 2b_2 + b_1 = 0$ $\Rightarrow$ Infinitely many solution

$a_1 = -3b_1 + 2b_2 + a_3$

$a_2 = b_1 - \frac{b_1}{2} - 2a_3$

$$V = \{b \mid b \in \mathbb{R}^3, \ b = a_1v_1 + a_2v_2 + a_3v_3\} = \left\{ \begin{pmatrix} b_1 \\ b_2 \\ 2b_2 - b_1 \end{pmatrix} : b_1, b_2 \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$$

$\| $

$\operatorname{span}\{v_1 \ v_2 \ v_3\}$

Check that $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ are linearly independent? Yes

$\Rightarrow$ $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ form a basis for V

$\dim V = 2$

Def. Standard basis (or canonical basis) for $\mathbb{R}^n$

is $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{-th}, \cdots \ e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

Check: ① $\operatorname{span}\{e_1, e_2, \cdots, e_n\} = \mathbb{R}^n$

② $\{e_1, e_2, \cdots, e_n\}$ are linearly independent

$\Rightarrow$ It forms a basis

Note. Given $\{v_1, v_2, \cdots, v_n\}$, we construct $A = [v_1 \ v_2 \ \cdots v_n]$

① $\operatorname{span}\{v_1, \cdots, v_n\} = \operatorname{span}\{\text{pivot columns of } A\}$

② $\dim \operatorname{span}\{v_1, \cdots, v_n\} = \operatorname{rank}(A)$

Usually, $\operatorname{span}\{v_1, \cdots, v_n\} = \operatorname{col}(A)$