§3.5 Vector spaces and Subspaces

A vector space is a nonempty collection of objects called vectors for which are defined the operations

- vector addition, denoted $\vec{x} + \vec{y}$ and
- scalar multiplication, denoted $c\vec{x}$

that satisfy the following properties for all $\vec{x}, \vec{y}, \vec{z} \in V$ and $c, d \in \mathbb{R}$ ($\mathbb{C}$)

Closure Properties:

① $\vec{x} + \vec{y} \in V$

② $c\vec{x} \in V$

$\Longleftrightarrow$ $c\vec{x} + d\vec{y} \in V \quad \forall\, \vec{x}, \vec{y} \in V$, $c, d \in \mathbb{R}$

Addition properties.

③ There is a zero vector $\vec{0}$ in V : $\vec{x} + \vec{0} = \vec{x} \quad \forall\, \vec{x} \in V$

④ Every vector $\vec{x}$ has its negative: $\vec{x} + (-\vec{x}) = 0$

⑤ $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$

⑥ $\vec{x} + \vec{y} = \vec{y} + \vec{x}$

Scalar Multiplication properties:

⑦ $1 \cdot \vec{x} = \vec{x}$

⑧ $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$

⑨ $(c+d) \cdot \vec{x} = c\vec{x} + d\vec{x}$

⑩ $c(d\vec{x}) = (cd)\vec{x}$

Ex: $V = \mathbb{R}$ is a vector space over $\mathbb{R}$ with the standard addition and multiplication.

The vectors are simply the real numbers

$\mathbb{R}^n$

Ex: $V = \mathbb{R}$ is <u>NOT</u> a vector space over $\mathbb{C}$ because $\mathbb{R}$ is not closed under multiplication by scalars ($\mathbb{C}$)

Ex: $M_{mn} = \{\text{All } m \times n \text{ real matrices}\} = \mathbb{R}^{m \times n}$ is a vector space over $\mathbb{R}$ with the matrix

Addition and ~~Matrix~~ Scalar Multiplication.
The vectors are $m \times n$ matrices

Ex: $\mathbb{C}^n$ is a vector space over $\mathbb{C}$ and over $\mathbb{R}$
The vectors are n-dimensional complex vectors

Ex: Consider the system of equations

$$\left.\begin{aligned} x + y &= 0 \\ 2x - y &= 0 \end{aligned}\right\} \quad \text{or} \quad A\vec{x} = \vec{b} \quad A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

V is a vector space over $\mathbb{R}$

Closure: $\forall\, \vec{x}_1, \vec{x}_2 \in V$, check $\vec{x}_1 + \vec{x}_2 \in V$ ✓

$$A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \vec{0} + \vec{0} = \vec{0}$$

$\forall\, \vec{x}_1 \in V$, $c \in \mathbb{R}$, check $c\vec{x}_1 \in V$ ✓

$$A(c\vec{x}_1) = (Ac)\vec{x}_1 = (cA)\vec{x}_1$$
$$= c(A\vec{x}_1) = c\vec{0} = \vec{0}$$

Properties (3) – (10) are satisfied ~~...~~ in $\mathbb{R}^2$

<u>Note</u>: $V = \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = 0 \}$ where $A \in \mathbb{R}^{m \times n}$
V is a vector space over $\mathbb{R}$

<u>Ex</u>: $V = \{ f : \mathbb{R} \to \mathbb{R} \mid f$ is <u>continuously differentiable</u> $\}$
V is a vector space

If $a, b \in \mathbb{R}$, $f, g \in V$, check $af + bg \in V$

i)
$$\left.\begin{aligned} &① \text{ If } f, g \text{ cont} \Rightarrow f + g \text{ cont.} \\ &② \text{ If } f \text{ cont}, a \in \mathbb{R} \Rightarrow af \text{ cont.} \end{aligned}\right\} \Rightarrow af + bg \text{ cont.}$$

Note: f cont. means that if $x_n \to a \Rightarrow f(x_n) \to f(a)$

$$\text{If } x_n \to a \Rightarrow \left.\begin{aligned} f(x_n) &\to f(a) \\ g(x_n) &\to g(a) \end{aligned}\right\} \quad (f+g)(x_n) = f(x_n) + g(x_n)$$

$$\longrightarrow f(a) + g(a) \Rightarrow f+g \text{ cont.}$$

ii) If f, g are differentiable, is $f+g$ diff? Yes

$a \in \mathbb{R}$ $\quad (af)' = af'$

$(af+bg)' = (af)' + (bg)' = af' + bg'$

$\longrightarrow$ $af + bg$ is diff.

i) ii) $\rightarrow$ $af + bg \in V$

The space is usually written as $C^1(\mathbb{R})$

Vector Subspace Theorem:

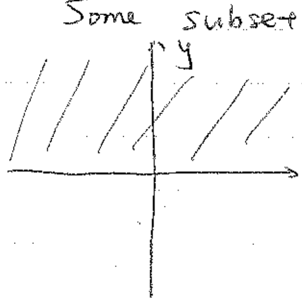
A nonempty subset W of a vector space V is a subspace of V if it is closed under addition and scalar multiplication:

$$\begin{cases} ① \ \vec{u} + \vec{v} \in W & \text{if } \vec{u}, \vec{v} \in W \\ ② \ c\vec{u} \in W & \text{if } \vec{u} \in W, \ c \in \mathbb{R} \end{cases}$$

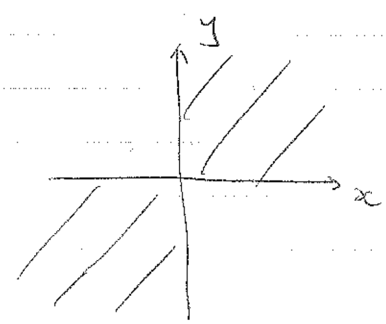
$\iff$ $a\vec{u} + b\vec{v} \in W$ if $\vec{u}\ \vec{v} \in W$, $a, b \in \mathbb{R}$

Note: $\vec{0} \in W$

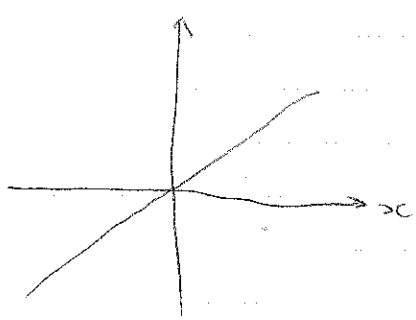
Ex: Some subset of $\mathbb{R}^2$



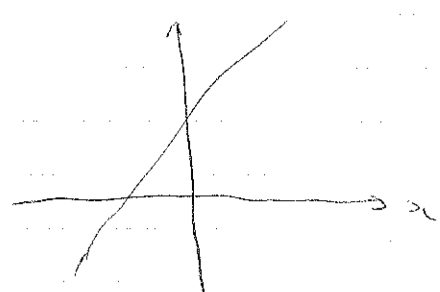
(a) Upper half-plane



(b) Quadrants I & III



(c) Line $y = x$ through the origin



(d) Line $y = x + 1$

(a) $M = \{(x,y) \mid y \geq 0\}$ is NOT a subspace of $\mathbb{R}^2$

M is not closed under scalar multiplication

$(0,1) \in M$ $\quad (-1)\cdot(0,1) = (0,-1) \notin M$

(b) $M = \{(x,y) \mid xy \geq 0\}$ is NOT a subspace of $\mathbb{R}^2$
because it is not closed under addition

$(2,1) \in M \quad (-1,-2) \in M$

$(2,1) + (-1,-2) = (1,-1) \notin M$

(c) $M = \{(x,y) \mid x = y\}$ is a subspace of $\mathbb{R}^2$

$\forall\ (x_1,y_1),\ (x_2,y_2) \in M \quad a,b \in \mathbb{R}$

$$a(x_1,y_1) + b(x_2,y_2) = (ax_1, ay_1) + (bx_2, by_2)$$
$$= (ax_1 + bx_2,\ ay_1 + by_2)$$
$$= (ax_1 + bx_2,\ ax_1 + bx_2) \in M$$

(d) $M = \{(x,y) \mid y = x+1\}$ is NOT is a subspace of $\mathbb{R}^2$
because $(0,0) \notin M$

Subspaces of $\mathbb{R}^2$:

- zero subspace $M = \{(0,0)\}$ > trivial subspace
- $\mathbb{R}^2$ itself
- lines passing through the origin

Ex: $M = \{\vec{x} \mid A\vec{x} = \vec{0}.\ A \in \mathbb{R}^{m\times n}\}$ is a subspace of M_{mn}

$\forall\ \vec{x}, \vec{y} \in M.\quad a,b \in \mathbb{R} \quad$ check $a\vec{x} + b\vec{y} \in M$

$$A(a\vec{x} + b\vec{y}) = A(a\vec{x}) + A(b\vec{y})$$
$$= (Aa)\vec{x} + (Ab)\vec{y}$$
$$= (aA)\vec{x} + (bA)\vec{y}$$
$$= a(A\vec{x}) + b(A\vec{y})$$
$$= 0$$

Ex: $M = \{f \mid f \in C([0,1]),\ f(0) = 0\}$ is a subspace of $C([0,1])$

$\forall\ f, g \in M,\quad a, b \in \mathbb{R}$ check $af + bg \in M$

f cont $\rightarrow$ af cont. so is bg

$\rightarrow af + bg$ is cont.

$$(af+bg)(0) = (af)(0) + (bg)(0)$$
$$= a f(0) + b g(0)$$
$$= a\cdot 0 + b \cdot 0 = 0$$