§ 3.4 Determinants and Cramer's Rule

Consider a 2×2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

We define the determinant of A

$$\det(A) = |A| = a_{11} \cdot a_{22} - a_{12} a_{21} \leftarrow \text{scalar}$$

Ex: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \rightarrow |A| = 1 \cdot 4 - 2 \cdot 3 = -2$

$A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \rightarrow |A| = 1 \cdot 4 - 3 \cdot 2 = -2$

<u>Note</u>: $A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \rightarrow |A^T| = a_{11} a_{22} - a_{21} \cdot a_{12} = |A|$

We are going to generalize the ~~product~~ concept of determinat to square matrices of any size

Def: Given an $n \times n$ matrix A we define

* the (i,j) - minor as

M_{ij} = matrix obtained from A by removing row i and column j an $(n-1) \times (n-1)$ matrix

* the cofactor of a_{ij}

$$C_{ij} = (-1)^{i+j} |M_{ij}| \quad \text{scalar}$$

Def: Given an $n \times n$ matrix A, we compute the determinant by

$$|A| = \sum_{j=1}^{n} a_{ij} \cdot C_{ij} \quad \text{for any } i \quad (i \text{ is fixed})$$

(expanding by row i)

or $$|A| = \sum_{i=1}^{n} a_{ij} C_{ij} \quad \text{for any } j \quad (j \text{ is fixed})$$

(expanding by column j)

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ $|A| = ?$

Minors: $M_{11} = \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix}$ $M_{12} = \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix}$ $M_{13} = \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$

$M_{21} = \begin{pmatrix} 2 & 3 \\ 8 & 9 \end{pmatrix}$ $M_{22} = \begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix}$ $M_{23} = \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix}$

$M_{31} = \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix}$ $M_{32} = \begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix}$ $M_{33} = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$

Cofactor $C_{11} = (-1)^{1+1} \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = 5 \cdot 9 - 6 \cdot 8 = -3$

$C_{12} = (-1)^{1+2} |M_{12}| = -(4 \cdot 9 - 6 \times 7) = 6$

$C_{13} = (-1)^{1+3} |M_{13}| = 4 \times 8 - 5 \times 7 = -3$

$C_{21} = 6 \quad C_{22} = -12 \quad C_{23} = 6$

$C_{31} = -3 \quad C_{32} = 6 \quad C_{33} = -3$

$|A| = a_{11} \cdot C_{11} + a_{12} C_{12} + a_{13} C_{13}$ $\qquad i = 1$

$= 1 \cdot (-3) + 2 \cdot 6 + 3 \cdot (-3) = 0$

$|A| = a_{21} C_{21} + a_{22} C_{22} + a_{23} C_{23}$ $\qquad i = 2$

$= 4 \cdot 6 + 5(-12) + 6 \cdot 6 = 0$

$|A| = a_{13} C_{13} + a_{23} C_{23} + a_{33} C_{33}$ $\qquad j = 3$

$= 3 \cdot (-3) + 6 \cdot 6 + 9 \cdot (-3) = 0$

Ex: $A = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & -1 & 4 \\ 2 & 0 & 3 & 0 \end{pmatrix}$ $\qquad |A| = ?$

We choose $j = 1$ (only two nonzeros)

$|A| = a_{11} C_{11} + a_{41} C_{41}$

$C_{11} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 0 & 3 & 0 \end{vmatrix}$ $\qquad C_{41} = -\begin{vmatrix} 0 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & -1 & 4 \end{vmatrix}$

Note: Basket weave technique for calculating determinant of 3×3 matrices

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

red : plus

green : minus

$$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{22}a_{32}$$
$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

$$C_{11} = 3 \cdot (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 6$$

$$C_{41} = -1 \cdot 1 \cdot (-1)^{2+1} \begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix} - 1 \cdot 2 \cdot (-1)^{3+1} \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix}$$
$$= 11$$

$$|A| = 1 \cdot 6 + 2 \cdot 11 = 28$$

Properties of determinants:

① If two columns (rows) are interchanged in A to obtain matrix B, then $|B| = -|A|$

2x2 matrices.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad B = \begin{pmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{pmatrix}$$

$$\det(B) = a_{12}a_{21} - a_{11}a_{22} = -(a_{11}a_{22} - a_{12}a_{21}) = -|A|$$

② If one column (or row) is multiplied by a constant k and added to another column (row), the determinant does not change

2x2 case:

$$B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21}+ka_{11} & a_{22}+ka_{12} \end{pmatrix}$$

$$|B| = a_{11}(a_{22}+ka_{12}) - a_{12}(a_{21}+ka_{11})$$
$$= a_{11}a_{22} + ka_{11}a_{12} - a_{12}a_{21} - ka_{12}a_{11}$$
$$= a_{11}a_{22} - a_{12}a_{21} = |A|$$

③ If we multiply a row of A (column) by a constant k then the determinant is multiplied by the same constant

2×2 case: $B = \begin{pmatrix} ka_{11} & ka_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$|B| = ka_{11}a_{22} - ka_{12}a_{21} = k(a_{11}a_{22} - a_{12}a_{21}) = k\,|A|$$

Note: $|kA| = k^n |A|$ if $A \in \mathbb{R}^{n\times n}$

(4) If two of the columns (rows) of A are the same, then $|A| = 0$

Pf: Assume i-th column is the same as j-th column

$B =$ matrix where we swap i-th and j-th column

$$\left.\begin{array}{l} \overset{(1)}{\Longrightarrow} |B| = -|A| \\ \text{but } B = A \quad |B| = |A| \end{array}\right\} \rightarrow |A| = 0$$

Theorem: A $n\times n$ square matrix is invertible if and only if $|A| \neq 0$

Proof: A invertible $\Longleftrightarrow$ A has n pivots Gaussian elimination

leads to $B = \begin{pmatrix} a_{11} & x & \cdots & x \\ & a_{22} & x \cdots & x \\ & 0 & \ddots & \vdots \\ & & & x \\ & & & a_{nn} \end{pmatrix}$ upper Triangular matrix

$$|B| = a_{11}\cdot C_{11} = a_{11} \begin{vmatrix} a_{22} & x & \cdots & x \\ & a_{33} & & \\ 0 & & \ddots & x \\ & & & a_{nn} \end{vmatrix} = a_{11}a_{22} \begin{vmatrix} a_{33} & x & - & x \\ & \ddots & & \vdots \\ 0 & & \ddots & x \\ & & & a_{nn} \end{vmatrix}$$

$$= a_{11}a_{22}\cdots a_{nn} \neq 0$$

Since they are pivots $\Rightarrow$ $\det A \neq 0$

Cramer's Rule:

Consider $Ax = b$ $n\times n$ system and assume $|A| \neq 0$

Define $A_i =$ matrix obtained from A by replacing the i-th column with b, then

The solution of $Ax = b$ is

$$x_i = \frac{|A_i|}{|A|}$$

Ex: $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $A\vec{x} = b$ $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$|A| = 1\cdot(-1) - 1\cdot 1 = -2 \neq 0$

① Gauss: $\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{2} \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \end{array}\right)$

② Cramer's Rule:

$$x = \frac{1}{|A|} \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = \frac{1}{-2}(-1-2) = \frac{3}{2}$$

$$y = \frac{1}{|A|} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = \frac{1}{-2}(2-1) = -\frac{1}{2}$$

Properties: ① $|A+B| \neq |A| + |B|$ in general

Ex: $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$|A| = 0$ $|B| = 0$ $|A+B| = 1$

② $|AB| = |A| \cdot |B|$

<u>Note</u>: $|BA| = |B| \cdot |A| = |A| \cdot |B| = |AB|$
even though $AB \neq BA$ in general

③ $|A^T| = |A|$

④ $|A^{-1}| = \frac{1}{|A|}$ since $A \cdot A^{-1} = I \Rightarrow |A \cdot A^{-1}| = 1 = |A||A^{-1}|$
$\Rightarrow |A^{-1}| = \frac{1}{|A|}$

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$A \to A^{-1}$ $\quad A x_i = e_i \quad x_i = A^{-1} e_i$

From Cramer's Rule, we have

$$x_i = \frac{1}{\det(A)} \begin{pmatrix} |A_1| \\ |A_2| \\ \vdots \\ |A_n| \end{pmatrix} = \frac{1}{\det(A)} \begin{pmatrix} C_{1i} \\ C_{2i} \\ \vdots \\ C_{3i} \end{pmatrix}$$

$$A^{-1} = (x_1 | x_2 | \cdots | x_n) = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ - & - & - & - \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}$$

C_{ij} (ij) - cofactor

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} \qquad \text{for any } i$$

$$\text{or} = \sum_{i=1}^{n} a_{ij} C_{ij} \qquad \text{for any } j$$