§3.3 The Inverse of a Matrix

Image we want to solve $ax = b$

$x = \frac{b}{a}$ or $a^{-1} \cdot b$ if $a \neq 0$

or Inverse of a number ($\neq 0$)

$a \cdot b = b \cdot a = 1$ $\quad (b = a^{-1} = \frac{1}{a})$

Now we generalize this to matrices

Def: We say that a square matrix A is invertable (or it has an inverse) if there exist a square matrix B (with the same dimensions) such that

$$A \cdot B = B \cdot A = I$$

We denote the inverse by A^{-1} $(= B)$

Note: Not every nonzero matrix has an inverse.

Assume that A is invertible. We want to solve $A\vec{x} = \vec{b}$

$A^{-1} \cdot (A\vec{x}) = A^{-1} \cdot \vec{b}$

LHS $= (A^{-1}A)\vec{x} = A^{-1} \cdot \vec{b} \rightarrow \vec{x} = A^{-1}\vec{b}$

LHS $= I\vec{x} = \vec{x}$

<u>Note</u>: $Ax = b$ has a unique solution if A is invertible

$x = A^{-1} b$

* Inverse Matrix by Gaussian Elimination

Assume A is invertible, then $\exists B$, such that $A \cdot B = I$

Or

$$A \cdot [b_1, b_2 \cdots b_n] = [e_1 \; e_2 \cdots e_n]$$

where $B = [b_1 \; b_2 \cdots b_n]$ $\quad b_i \; i = 1, 2, \cdots, n$ ~~i-th~~ column vector of B

$I = [e_1 \; e_2 \cdots e_n]$ $\quad e_i$ $\quad$ i-th column vector of I

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{i-th row}$$

$$A \cdot B = [Ab_1 \quad Ab_2 \quad \cdots \quad Ab_n] = [e_1 \quad e_2 \quad \cdots \quad e_n]$$

$$\rightarrow \quad Ab_1 = e_1 \quad Ab_2 = e_2 \quad \cdots \quad Ab_n = e_n$$

We can get B by solving n-systems of equations

Ex: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\quad A^{-1} = ?$

We need to solve $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ first column of A^{-1}

$$\left.\begin{matrix} x + 2y = 1 \\ 3x + 4y = 0 \end{matrix}\right\} \rightarrow \left.\begin{matrix} x + 2y = 1 \\ -2y = -3 \end{matrix}\right\} \rightarrow \left.\begin{matrix} x = -2 \\ y = \frac{3}{2} \end{matrix}\right\} \quad b_1 = \begin{pmatrix} -2 \\ \frac{3}{2} \end{pmatrix}$$

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ second column of A^{-1}

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -2 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc} 1 & 2 & 0 \\ 0 & 1 & -\frac{1}{2} \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} \end{array}\right) \quad b_2 = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

Verification: $AA^{-1} = A^{-1}A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

We can solve systems of equations simultaneously

$$Ab_1 = c_1 \quad Ab_2 = c_2 \quad \cdots \quad Ab_m = c_m$$

Define the augumented matrix

$$\bar{A} = \left(A \mid c_1 \mid c_2 \mid \cdots \mid c_m \right) \Rightarrow \text{Solve like before}$$

$$\bar{A} = \left(\begin{array}{cc|c|c} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right) \xrightarrow{R_2^* = R_2 - 3R_1} \left(\begin{array}{cc|c|c} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right) \xrightarrow{R_2^* = -\frac{1}{2}R_2}$$

$$\left(\begin{array}{cc|c|c} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right) \xrightarrow{R_1^* = R_1 - 2R_2} \left(\begin{array}{cc|c|c} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right)$$

$$\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I} \qquad \underbrace{\begin{matrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{matrix}}_{A^{-1}}$$

$$A^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

A is row equivalent to I_n

Theorem: Given A $n \times n$ invertible matrix and B such that

$$A \cdot B = I \quad (\text{or } (B \cdot A) = I) \Longrightarrow B = A^{-1}$$

Proof: Assume $A \cdot B = I \Rightarrow A^{-1}(A \cdot B) = A^{-1} \cdot I$
$\Rightarrow (A^{-1}A) \cdot B = A^{-1} \cdot I \Rightarrow I \cdot B = A^{-1} \Rightarrow B = A^{-1}$

Corollary: The Inverse of a matrix is unique

Note: Not every nonzero matrix has an inverse
Ex: $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 4 & 0 \end{array}\right] \xrightarrow{R_2^* = R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & -2 \end{array}\right] \to 0 = -2$$

There is no solution $\Rightarrow$ A does not have an inverse!

Theorem: Given A $n \times n$ matrix, the following statements are equivalent:
1. A is an invertible matrix ***
2. A^T is an invertible matrix
3. A is row equivalent to I_n
4. A has n pivot columns (rank A $= n$) ***
5. The system of equations $A\vec{x} = 0$ has only the zero solution ***
6. The system of equations $A\vec{x} = \vec{b}$ has a unique solution ***

Ex: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\quad Ax = 0$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

Zero solution only $\to$ A has an inverse

$$B = \begin{pmatrix} 1 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & -1 & 1 \end{pmatrix} \qquad B^{-1} = ?$$

$$\bar{B} = \left(\begin{array}{ccc|ccc} 1 & -3 & 0 & 1 & 0 & 0 \\ 2 & -5 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2^* = R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & -3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3^* = R_3 + R_2} \left(\begin{array}{ccc|ccc} 1 & -3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right) \xrightarrow{R_1^* = R_1 + 3R_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 3 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right)$$

$$B^{-1} = \begin{pmatrix} -5 & 3 & 0 \\ -2 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix}$$