§3.2 Systems of Linear Equations

A linear system of equations is a system of the form

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} (*)$$

m - equations with n unknows $x_1, x_2, \cdots, x_n$
(in n variables)

We want to find $x_1, x_2, \cdots, x_n$ that will satisfy $(*)$

Every linear system can be written in the form $Ax = b$
where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{R}^{m\times n}$ $\quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ $\quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \in$

Homogeneous $\Leftrightarrow$ $\vec{b} = 0$

<u>Ex:</u> $x + y = 0$ In matrix form $(1, 1)\cdot\begin{pmatrix} x \\ y \end{pmatrix} = 0$

If $x + y = 0 \Rightarrow y = -x$

$\Rightarrow$ any pair of the form $(x, -x)$ is a solution

<u>Ex</u>: $x + y = 0$
$x - y = 2$

Substitution: If $x + y = 0 \Rightarrow y = -x$

$2 = x - y = x - (-x) = 2x \Rightarrow \begin{matrix} x = 1 \\ y = -1 \end{matrix}$

<u>Ex</u>:
$$\left.\begin{aligned} x + y &= 0 \\ x + 3y &= 2 \\ 2x + 4y &= 6 \end{aligned}\right\} \begin{aligned} &\rightarrow y = -x \\ &\longrightarrow x - 3x = 2 \Rightarrow -2x = 2 \\ &\longrightarrow 2x - 4x = 6 \Rightarrow -2x = 6 \end{aligned}$$

$\Longrightarrow$ The system has no solution

Gaussian Elimination (or Gauss - Jordan elimination)

Look at the coefficients of x_1. Assume $a_{11} \neq 0$

Fix the first eqn: $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$

Multiply the first eqn by $\dfrac{a_{21}}{a_{11}}$

Replace the second equation by $R_2^* = R_2 - \frac{a_{21}}{a_{11}} R_1$
(equation)

R_i denotes the i-th row of the matrix before the operation is applied

R_i^* denotes the i-th row after the operation has been applied

Elementary Row Operations

① Interchange row i and row j:

$$R_i \Leftrightarrow R_j \quad (\text{or } R_i^* = R_j,\ R_j^* = R_i)$$

② Multiply row i by a constant $c \neq 0$:

$$R_i^* = c R_i$$

③ Add c time row j to row i (leaving row j unchanged)

$$R_i^* = R_i + c R_j \qquad (R_j^* = R_j)$$

$$\left(a_{22} - \frac{a_{21}}{a_{11}} a_{12}\right) x_2 + \left(a_{23} - \frac{a_{21}}{a_{11}} a_{13}\right) x_3 + \cdots = b_2 - \frac{a_{21}}{a_{11}} b_1$$

Repeat: $R_3^* = R_3 - \frac{a_{31}}{a_{11}} R_1$

$$\vdots$$

$$R_m^* = R_m - \frac{a_{m1}}{a_{11}} R_1$$

Now the new system looks like this

$$\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= b_1 \\
a_{22}^* x_2 + \cdots + a_{2n}^* x_n &= b_2^* \\
a_{32}^* x_2 + \cdots + a_{3n}^* x_n &= b_3^* \\
&\vdots \\
a_{m2}^* x_2 + \cdots + a_{mn}^* x_n &= b_m^*
\end{aligned}$$

If we iterate, we end up with

$$\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= b_1 \\
a_{22}^* x_2 + \cdots + a_{2n}^* x_n &= b_2^* \\
\tilde{a}_{33} x_3 + \cdots + \tilde{a}_{3n} x_n &= \tilde{b}_3 \\
&\ddots \\
\tilde{\tilde{a}}_{mm} x_m + \cdots + \tilde{\tilde{a}}_{mn} x_n &= \tilde{\tilde{b}}_m
\end{aligned}$$

Assume $m = n$ then the last equation is

$$\tilde{\tilde{a}}_{nn} x_n = \tilde{\tilde{b}}_n \Rightarrow x_n = \frac{\tilde{\tilde{b}}_n}{\tilde{\tilde{a}}_{nn}} \quad (\text{if } \tilde{\tilde{a}}_{nn} \neq 0)$$

Then $(n-1)$ eqn is

$$\tilde{a}_{n-1,n-1} x_{n-1} + \tilde{a}_{n-1,n} x_n = \tilde{b}_{n-1}$$

If $\widetilde{a}_{n-1,n-1} \neq 0 \Rightarrow x_{n-1} = \dfrac{\widetilde{b}_{n-1} - \widetilde{a}_{n-1,n} x_n}{\widetilde{a}_{n-1,n-1}}$

This process is called <u>backward substitution</u>

<u>Ex</u>: $\left.\begin{aligned} x + y + z &= 3 \\ 2x - 3y - z &= -8 \\ -x + 2y + 2z &= 3 \end{aligned}\right\}$ $\begin{aligned} x + y + z &= 3 \\ 5y + 3z &= 14 \quad & R_2^* = -R_2 + 2R_1 \\ 3y + 3z &= 6 \quad & R_3^* = R_3 + R_1 \end{aligned}$

$$\begin{aligned} x + y + z &= 3 \\ 5y + 3z &= 14 \\ 6z &= -12 \quad & R_3^* = 5R_3 - 3R_2 \end{aligned}$$

Note: $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & 6 \end{pmatrix}$ upper triangular

Backward Substitution. $z = -2 \quad y = \dfrac{14 - 3z}{5} = 4$

$x = 3 - y - z = 1$

Verification: $x + y + z = 1 + 4 - 2 = 3$ ✓

$2x - 3y - z = 2 - 3 \cdot 4 - (-2) = -8$ ✓

$-x + 2y + 2z = -1 + 2 \cdot 4 + 2 \cdot (-2) = 3$ ✓

<u>Ex</u>. $\left.\begin{aligned} x + 2y + 3z + w &= 0 \\ x - 2y + z - w &= 0 \\ x + y &= 0 \end{aligned}\right\}$

$\left.\begin{aligned} x + 2y + 3z + w &= 0 \\ 4y + 2z + 2w &= 0 \\ y + 3z + w &= 0 \end{aligned}\right\}$ $\begin{aligned} x + 2y + 3z + w &= 0 \\ y + 3z + w &= 0 \\ -10z - 2w &= 0 \end{aligned}$

Set w as a parameter

$z = \dfrac{2w}{-10} = -\dfrac{w}{5}$

$y = -w - 3z = -w - 3 \cdot \left(-\dfrac{w}{5}\right) = -\dfrac{2}{5} w$

$x = -2y - 3z - w = (-2) \cdot \left(-\dfrac{2}{5} w\right) - 3 \cdot \left(-\dfrac{w}{5}\right) - w = \dfrac{2}{5} w$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = w \begin{pmatrix} \frac{2}{5} \\ -\frac{2}{5} \\ -\frac{1}{5} \end{pmatrix} \Rightarrow \text{Infinitely many solutions}$$

<u>Ex</u>.

$$\left.\begin{aligned} x+y &= 1 \\ x-y &= 2 \\ 2x+3y &= 10 \end{aligned}\right\} \quad \left.\begin{aligned} x+y &= 1 \\ 2y &= -1 \\ y &= 8 \end{aligned}\right\} \quad \begin{aligned} x+y &= 1 \\ 2y &= -1 \\ 0 &= 17 \quad ?? \end{aligned}$$

Going back to the original problem:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$\ddots$$

$$a_{mm}x_m + \cdots + a_{mn}x_n = b_m \qquad \left\{\begin{array}{l} a_{11} \neq 0 \\ a_{22} \neq 0 \\ \vdots \\ a_{mm} \neq 0 \end{array}\right.$$

$$0x_1 + \cdots + 0x_n = b_{m+1}$$

$$\vdots$$

$$0x_1 + \cdots + 0x_n = b_n$$

For us to have a solution, (for the system to be consistent) we need

$$b_{m+1} = b_{m+2} = \cdots = b_n = 0$$

Then the solution is

$$x_m = \frac{\tilde{b}_m}{a_{mm}} + \alpha_{m1} x_{m+1} + \alpha_{m2} x_{m+2} + \cdots + \alpha_{m,n-m} x_n$$

$$x_{m-1} = \frac{\tilde{b}_{m-1}}{a_{m-1\,m-1}} + \alpha_{m-1,1} x_{m+1} + \alpha_{m-1,2} x_{m+2} + \cdots + \alpha_{m-1,n-m} x_n$$

$$\vdots$$

$$x_1 = \frac{\tilde{b}_1}{a_{11}} + \alpha_{11} x_{m+1} + \cdots + \alpha_{1,n-m} x_n$$

The solution looks like this:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \text{particular solution} + x_{m+1}\begin{pmatrix} \\ \\ \end{pmatrix} + x_{m+2}\begin{pmatrix} \\ \\ \end{pmatrix} + \cdots \; x_n\begin{pmatrix} \\ \\ \end{pmatrix}$$

Summary: ① Do Gauss elimination and reduce the problem to an upper triangular matrix

Def: If the diagonal coefficient in the reduced system is not zero, that eqn is called pivot.

Def. The Rank of a matrix = # pivots

Note: I will assume the # equations and # unknowns is the same: $m = n$

We reduce the system to upper triangular

Count the number of pivots p

① If $p = n \Rightarrow$ Unique solution

(Solve it by backward substitution)

The matrix has full rank

② If $p < n \Rightarrow$ two possibilities

i) If all the remaining equations are of the form

$0x_1 + \cdots + 0x_n = 0 \leftarrow$ (system is consistent)

$\Rightarrow$ Infinitely many solutions of the form

$$\vec{z} = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} = \vec{z_0} + x_{p+1}\vec{v_1} + x_{p+2}\vec{v_2} + \cdots + x_n \vec{v}_{n-p}$$

ii) Exist at least one equation of the form

$0x_1 + \cdots + 0x_n = b \neq 0$

The system is inconsistent $\Rightarrow$ There are no solutions

Alternatively: $Ax = b$ system of eqns

Define the augmented matrix

$$\bar{A} = \left[A \mid b \right]$$

Gaussian elimination is equivalent to operating on rows of $\bar{A}$

Gaussian elimination leads to a matrix of the form

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ & a_{22} & \cdots & a_{2n} & b_2 \\ & & \ddots & & \vdots \\ & & a_{qq} \cdots & a_{qn} & b_q \\ \hline & & 0 & & b^* \end{array}\right]$$

q = rank
= # pivots

If $q = n \Rightarrow$ unique solution

$q < n \Rightarrow \begin{cases} b^* = 0 \rightarrow \text{infinitely many sol.} \\ b^* \neq 0 \Rightarrow \text{no sol} \end{cases}$

If we further do backward substitution

$$\left[\begin{array}{cccc|ccc|c} a_{11} & 0 & \cdots & 0 & a_{1q+1} & \cdots & a_{1n} & b_1 \\ & a_{22} & 0 \cdots & 0 & a_{2q+1} & \cdots & a_{2n} & b_2 \\ 0 & & \ddots & & & \vdots & & \vdots \\ & & & a_{qq} & a_{qq+1} & \cdots & a_{qn} & b_q \\ \hline & 0 & & & & 0 & & b^* \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc|c} 1 & & 0 & \tilde{a}_{1q+1} & \cdots & \tilde{a}_{1n} & \tilde{b}_1 \\ & 1 & & \tilde{a}_{2q+1} & \cdots & \tilde{a}_{2n} & \tilde{b}_2 \\ 0 & & \ddots & & \cdots & & \vdots \\ & & 1 & \tilde{a}_{qq+1} & \cdots & \tilde{a}_{qn} & \tilde{b}_q \\ \hline & 0 & & & 0 & & b^* \end{array}\right]$$

which is called reduced row echelon form (RREF)

Ex: $x + y + z = 0$ 1 pivot $\Rightarrow$ 2 parameters

$x = -y - z$

Ex: $x + 2y + 3z + 4w = 1$

$x - y - 2z + w = 2$

$2x + 4y - z + 3w = 3$

$3x + 4y - z = 4$

Augmented Matrix

$$\bar{A} = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 1 & -1 & -2 & 1 & 2 \\ 2 & 4 & -1 & 3 & 3 \\ 3 & 4 & -1 & 0 & 4 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 0 & 3 & 5 & 3 & -1 \\ 0 & 0 & -7 & -5 & 1 \\ 0 & -2 & -10 & -12 & 1 \end{array}\right]$$

$R_1^* = R_1$
$R_2^* = -R_2 + R_1$
$R_3^* = R_3 - 2R_1$
$R_4^* = R_4 - 3R_1$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 0 & 3 & 5 & 3 & -1 \\ 0 & 0 & -7 & -5 & 1 \\ 0 & 0 & -20 & -30 & 1 \end{array}\right] \quad \begin{array}{l} R_1^* = R_1 \\ R_2^* = R_2 \\ R_3^* = R_3 \\ R_4^* = 3R_4 + 2R_2 \end{array} \quad \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 0 & 3 & 5 & 3 & -1 \\ 0 & 0 & -7 & -5 & 1 \\ 0 & 0 & 0 & -110 & -13 \end{array}\right] \quad R_4^* = 7R_4 - 20R_3$$

$\Longrightarrow$ # pivots $= 4 =$ # eqn $=$ # unknowns

$\longrightarrow \exists !$ solution

~~there~~ exists unique

$$\begin{cases} w = \frac{13}{110} \\ z = -\frac{1}{7}(1+5w) = -\frac{1}{7}\left(1+\frac{13}{22}\right) = -\frac{5}{22} \\ y = \frac{1}{3}(-1-3w-5z) = \frac{1}{3}\left(-1-\frac{39}{110}+\frac{25}{22}\right) = -\frac{4}{55} \\ x = 1-2y-3z-4w = 1+\frac{8}{55}+\frac{15}{22}-\frac{26}{55} = \frac{149}{110} \end{cases}$$

Backward Substitution

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 0 & 3 & 5 & 3 & -1 \\ 0 & 0 & -7 & -5 & 1 \\ 0 & 0 & 0 & 1 & \frac{13}{110} \end{array}\right] \quad R_4^* = -\frac{1}{110}R_4 \quad \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 0 & 3 & 5 & 3 & -1 \\ 0 & 0 & 1 & 0 & -\frac{5}{22} \\ 0 & 0 & 0 & 1 & \frac{13}{110} \end{array}\right] \quad R_3^* = -\frac{1}{7}(R_3 + 5R_4)$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 0 & 1 & 0 & 0 & -\frac{4}{55} \\ 0 & 0 & 1 & 0 & -\frac{5}{22} \\ 0 & 0 & 0 & 1 & \frac{13}{110} \end{array}\right] \quad R_2^* = \frac{1}{3}(R_2 - 5R_3 - 3R_4) \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{149}{110} \\ 0 & 1 & 0 & 0 & -\frac{4}{55} \\ 0 & 0 & 1 & 0 & -\frac{5}{22} \\ 0 & 0 & 0 & 1 & \frac{13}{110} \end{array}\right] \quad R_1^* = R_1 - 2R_2 - 3R_3 - 4R_4$$

RREF

Ex: $x + y = 0$, $-x + ay = 1$ $\quad \begin{cases} x + y = 0 \\ (a+1)y = 1 \end{cases}$

$\rightarrow$ Unique solution $\Longleftrightarrow a+1 \neq 0$ i.e., $a = -1$

If $a = -1$ $\quad \begin{array}{l} x + y = 0 \\ 0 = 1 \end{array}$ # No solution

Ex: $x + y = 0$, $-x + ay = b$ $\quad \begin{array}{l} x + y = 0 \\ (a+1)y = b \end{array}$

If $a+1 \neq 0 \Rightarrow y = \frac{b}{a+1} \quad x = -\frac{b}{a+1}$

If $a = -1 \Rightarrow \begin{array}{l} x + y = 0 \\ 0 = b \end{array}$

There is solution $\Leftrightarrow$ $b=0$
$\begin{pmatrix} x \\ -x \end{pmatrix}$

If $b \neq 0 \Rightarrow$ No solution

Ex

$$\left.\begin{aligned} 3x_1 + 5x_2 &= 1 \\ 3x_1 + 7x_2 + 3x_3 &= 8 \\ 5x_2 &= -5 \\ 2x_2 + 3x_3 &= 7 \\ x_1 + 4x_2 + x_3 &= 1 \end{aligned}\right\}$$

$\longrightarrow x_2 = -1$

$\longrightarrow x_3 = 3$

$\longrightarrow x_1 = 1 - 4x_2 - x_3 = 2$

$3x_1 + 5x_2 = 3\cdot 2 + 5\cdot(-1) = 1$ ✓

$3x_1 + 7x_2 + 3x_3 = 3\cdot 2 + 7\cdot(-1) + 3\cdot 3 = 8$ ✓

Augmented matrix

$$\left[\begin{array}{ccc|c} 3 & 5 & 0 & 1 \\ 3 & 7 & 3 & 8 \\ 0 & 5 & 0 & -5 \\ 0 & 2 & 3 & 7 \\ 1 & 4 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_2^* = R_2 - R_1 \\ \\ \\ \\ R_5^* = 3R_5 - R_1 \end{array} \qquad \left[\begin{array}{ccc|c} 3 & 5 & 0 & 1 \\ 0 & 2 & 3 & 7 \\ 0 & 5 & 0 & -5 \\ 0 & 2 & 3 & 7 \\ 0 & 7 & 3 & 2 \end{array}\right] \quad \begin{array}{l} R_2^* = \frac{1}{5}R_3 \\ R_3^* = R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 5 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 2 & 3 & 7 \\ 0 & 2 & 3 & 7 \\ 0 & 7 & 3 & 2 \end{array}\right] \quad \begin{array}{l} R_3^* = R_3 - 2R_2 \\ R_4^* = R_4 - 2R_2 \\ R_5^* = R_5 - 7R_2 \end{array} \qquad \left[\begin{array}{ccc|c} 3 & 5 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 3 & 9 \end{array}\right] \quad \begin{array}{l} R_3^* = \frac{1}{3}R_3 \\ R_4^* = R_4 - R_3 \\ R_5^* = R_5 - R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 5 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1^* = \frac{1}{3}(R_1 - 5R_2) \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{RREF}$$

Ex:

$$\left.\begin{aligned} x_1 + 2x_3 + x_5 + x_6 &= 8 \\ 2x_2 - 2x_4 - 4x_5 - 6x_6 &= 6 \\ x_3 + 2x_6 &= 2 \\ 3x_1 + x_4 + 5x_5 + 3x_6 &= 12 \end{aligned}\right\}$$

$$\left.\begin{aligned} x_1 + 2x_3 + x_5 + x_6 &= 8 \\ 2x_2 - 2x_4 - 4x_5 - 6x_6 &= 6 \\ x_3 + 2x_6 &= 2 \\ -6x_3 + x_4 + 2x_5 &= -12 \end{aligned}\right\}$$

$$\left.\begin{aligned} x_1 + 0x_2 + 2x_3 + 0x_4 + x_5 + x_6 &= 8 \\ 2x_2 + 0 - 2x_4 - 4x_5 - 6x_6 &= 6 \\ x_3 + 0x_4 + 0x_5 + 2x_6 &= 2 \\ x_4 + 2x_5 + 12x_6 &= 0 \end{aligned}\right\}$$

$$\Rightarrow x_4 = -2x_5 - 12x_6$$

$$x_3 = 2 - 2x_6$$

$$x_2 = \frac{1}{2}(6 + 2x_4 + 4x_5 + 6x_6) = 3 - 9x_6$$

$$x_1 = 8 - 2x_3 - x_5 - x_6 = 4 - x_5 + 3x_6$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 4 - x_5 + 3x_6 \\ 3 - 9x_6 \\ 2 - 2x_6 \\ -2x_5 - 12x_6 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} 3 \\ -9 \\ -2 \\ -12 \\ 0 \\ 1 \end{pmatrix}$$