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2011 Spring Math 3C

Differential Equations & Linear Algebra

(Chapters 1-3) Farlow et.al 2nd edition

Grading policy: homework (webwork) 40% mid-term exam 25% May 2

final exam 35% (June 10th) 8:00 -11:00 am

Next Friday

TA's office number, email & office hours

Office hour: MWF 9:00 - 10:00 am or by appointment

Enrld 27292 MWF 8:00 - 8:50am EMBAR HALL

68890 MWF 3:00 - 3:50pm HFH 1104

May 2
June 10
12 - 3pm

TA's office number, email & office hours

§3.1 Matrices: Sums and Products

A matrix is a rectangular array (2D) of elements or entries arranged in rows and columns

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Boldface Capital ~~letter~~

$A = [a_{ij}]$

lowercase letter with subscripts

Order: $m \times n$

Square matrix if $m = n$ order n

diagonal element: a_{ij} $(i = j)$

Column vector: $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$ $j = 1, \cdots, n$

Row vector: $[a_{i1}, a_{i2}, \cdots, a_{in}]$ $i = 1 \cdots m$

Ex $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

A is a 2×3 matrix or $A \in \mathbb{R}^{2\times3}$ — set of real numbers

$a_{11} = 1$ $a_{23} = 6$ ($\in$: belongs)

Column vectors: $\begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix} \in \mathbb{R}^{2\times1}$

Row vectors: $(1\ 2\ 3)$ $(4\ 5\ 6) \in \mathbb{R}^{1\times3}$

Def: Given $A \in \mathbb{R}^{m\times n}$ $B \in \mathbb{R}^{p\times q}$, we say that $A = B$ if $m = p$, $n = q$ and $a_{ij} = b_{ij}$ $i=1,\dots,m$ $j=1,\dots,n$

Ex: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$
not equal

Matrix Arithmetic

Def: (Addition of Matrices)
Given $A, B \in \mathbb{R}^{m\times n}$, then $C = A \pm B$ is the $\mathbb{R}^{m\times n}$ matrix given by $c_{ij} = a_{ij} \pm b_{ij}$ $\forall$ $i=1,\dots,m$, $j=1,\dots,n$
($\forall$: for all)

Ex: $A = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $A+B$ does not exist

Ex: $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \\ -1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 0 \\ -2 & 7 \\ 3 & 1/2 \end{pmatrix}$ $A+B = \begin{pmatrix} 1+2 & 2+0 \\ 3-2 & 2+7 \\ -1+3 & 0+1/2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 9 \\ 2 & \frac{1}{2} \end{pmatrix}$

Def: Multiplication by a Scalar
Given $A \in \mathbb{R}^{m\times n}$, and $c \in \mathbb{R}$, then $B = cA$ is the matrix of order $m\times n$ given by
$$b_{ij} = c\,a_{ij}$$

Ex: $A = \begin{pmatrix} -9 \\ 0 \\ 6 \end{pmatrix}$ $c = \pi$ $B = cA = \begin{pmatrix} -9\pi \\ 0 \\ 6\pi \end{pmatrix}$

Properties of Matrix Addition and Scalar Multiplication

Given $A, B, C \in \mathbb{R}^{m\times n}$, $c, k \in \mathbb{R}$

① $A+B$ is an $m\times n$ matrix (Closure)

② $A+B = B+A$ (Commutativity)

Proof: $(A+B)_{ij} = a_{ij}+b_{ij}$ $\Big\|$ they are equal for real numbers
$(B+A)_{ij} = b_{ij}+a_{ij}$

③ $A+(B+C) = (A+B)+C$ (Associativity)

Proof: $[A+(B+C)]_{ij} = a_{ij}+(B+C)_{ij} = a_{ij}+(b_{ij}+c_{ij}) = (a_{ij}+b_{ij})+c_{ij}$

$= (A+B)_{ij}+c_{ij} = [(A+B)+C]_{ij}$

④ $c(kA) = (ck)A$ (Associativity)

Proof: $[c(kA)]_{ij} = c(kA)_{ij} = c(k a_{ij}) = (ck)a_{ij} = (ck)[A]_{ij} = [(ck)A]_{ij}$

⑤ $A+0 = A$ (Zero Element)

Zero matric of size $m\times n$: $0_{m\times n}$ $(0)_{ij} = 0$ (zero)

⑥ $A+(-A) = 0$ (Inverse Element w.r.t. addition)

⑦ $c(A+B) = cA + cB$ (Distributivity)

⑧ $(c+k)A = cA + kA$ (Distributivity)

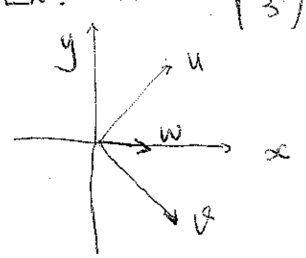
Def: Orthogonality: two vectors $u, v \in \mathbb{R}^n$ are said to be orthogonal if $u\cdot v = 0$

Def: Given two vectors $u, v \in \mathbb{R}^n$, we define their <u>scalar product</u> as

$$u\cdot v = u_1v_1 + u_2v_2 + \cdots + u_nv_n \quad \text{if } u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \quad v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

Ex: $u = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ $v = \begin{pmatrix} 4 \\ 1 \\ 1 \\ 2 \end{pmatrix}$ $u\cdot v = 1\cdot 4 + 2\cdot 1 + 0\cdot 1 + 1\cdot 2 = 8$

Ex: $u = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $v = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ $\Longrightarrow$ $u\cdot v = 0$



$w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\Longrightarrow$ $u\cdot w = 2$
$v\cdot w = 3$

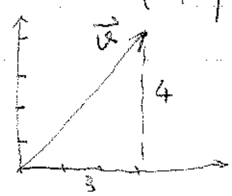
Def. Absolute value or norm of a vector

Given $v \in \mathbb{R}^n$, we define its norm as

$$\|v\|_2 = \sqrt{v \cdot v} = (v_1^2 + v_2^2 + \cdots + v_n^2)^{\frac{1}{2}} \quad (= \|v\|)$$

(Euclidian norm)

Ex. $\vec{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$



$\|v\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

$\equiv$ length of hypotenuse

$\|v\|^2 = 3^2 + 4^2 \longrightarrow$ Pythagoras

Def Product of Matrices, or Matrices Multiplication

Given $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$, we can define the product $A \cdot B$ only if $n = p$

We define the $m \times q$ matrix $A \cdot B$ given by

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2q} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nq} \end{pmatrix}$$

$C = A \cdot B$

$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$$

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} \qquad B = [\, b_1, b_2, \cdots, b_q \,]$$

$c_{ij} = a_i \cdot b_j \equiv$ i-th row of A $\cdot$ j-th column of B

Ex. $A = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 4 & 2 \end{pmatrix} \in \mathbb{R}^{2 \times 3}$ $\qquad B = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \in \mathbb{R}^{3 \times 1}$

$A \cdot B \in \mathbb{R}^{2 \times 1} \equiv \mathbb{R}^2$

scalar product

$(A \cdot B)_{11}$ = first row of A $\cdot$ first column of B

$= 1 \cdot 3 - 1 \cdot 2 - 1 \cdot 3 = -2$

$(A \cdot B)_{21}$ = second row of A $\cdot$ first column of B

$= 0 \cdot 3 + 4 \cdot 2 + 2 \cdot (-1) = 6$

$$A \cdot B = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$$

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \in \mathbb{R}^{2\times 3} \neq B = \begin{pmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \end{pmatrix} \in \mathbb{R}^{2\times 3}$

$\#(\text{Columns of } A) \neq \#(\text{Rows of } B)$

We cannot do the matrix product

Ex: $A = \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 3 \end{pmatrix} \in \mathbb{R}^{3\times 2}$ $\quad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \in \mathbb{R}^{2\times 2}$

$A \cdot B \in \mathbb{R}^{3\times 2}$ $\quad B \cdot A$ cannot be done

$(A\cdot B)_{11} = 1\cdot 1 + 0\cdot 2 = 1$ $\quad (A\cdot B)_{12} = 1\cdot 0 + 0\cdot 1 = 0$

$(A\cdot B)_{21} = 2\cdot 1 + 0\cdot 2 = 2$ $\quad (A\cdot B)_{22} = 2\cdot 0 + 0\cdot 1 = 0$

$(A\cdot B)_{31} = 1\cdot 1 + 3\cdot 2 = 7$ $\quad (A\cdot B)_{32} = 1\cdot 0 + 3\cdot 1 = 3$

$$A\cdot B = \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 7 & 3 \end{pmatrix}$$

Ex: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad AB \in \mathbb{R}^{2\times 2}$ $\quad BA \in \mathbb{R}^{2\times 2}$

$A\cdot B = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$ $\quad B\cdot A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$ $\quad A\cdot B \neq B\cdot A$

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \in \mathbb{R}^{2\times 3}$ $\quad B = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \in \mathbb{R}^{3\times 2}$

$AB \in \mathbb{R}^{2\times 2}$ $\quad BA \in \mathbb{R}^{3\times 3}$

Properties of Matrix Multiplication

① $(AB)C = A(BC)$ (Associativity)

Given $A \in \mathbb{R}^{m\times n}$ $\quad B \in \mathbb{R}^{n\times p}$ $\quad C \in \mathbb{R}^{p\times q}$

② $A(B+C) = AB + AC$ (Distributivity)

Given $A \in \mathbb{R}^{m\times n}$ $\quad B, C \in \mathbb{R}^{n\times p}$

③ $(B+C)A = BA + CA$ (Distributivity)

Given $B, C \in \mathbb{R}^{m\times n}$ $\quad A \in \mathbb{R}^{n\times p}$

④ $AB \neq BA$ except IN special cases Non commutativity

⑤ $AI_n = A$ $\quad I_m A = A$ $\quad A \in \mathbb{R}^{m\times n}$

Identity Matrix: $I_n \in \mathbb{R}^{n\times n}$ defined by $I_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

⑥ $A\, O_{np} = O_{mp}$ $\quad O_{qm} A = O_{qn}$ $\quad A \in \mathbb{R}^{m\times n}$

Def: Matrix Transpose

Given a matrix $A \in \mathbb{R}^{m\times n}$ we define its transpose A^T as the $n\times m$ matrix given by

$$(A^T)_{ij} = A_{ji}$$

Ex: $A = \begin{pmatrix} 2 & 1 \end{pmatrix} \in \mathbb{R}^{1\times 2}$ $\quad A^T = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \in \mathbb{R}^{2\times 1}$

Ex: $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 5 \end{pmatrix} \in \mathbb{R}^{2\times 3}$ $\quad A^T = \begin{pmatrix} 1 & 3 \\ 2 & 0 \\ -1 & 5 \end{pmatrix} \in \mathbb{R}^{3\times 2}$

Properties of Transpose:

① $(A^T)^T = A$ $\qquad A \in \mathbb{R}^{m\times n}$

② $(A+B)^T = A^T + B^T$ $\qquad A, B \in \mathbb{R}^{m\times n}$

③ $(kA)^T = kA^T$ $\qquad k \in \mathbb{R}$

④ $(AB)^T = B^T A^T$ $\qquad A \in \mathbb{R}^{m\times n},\ B \in \mathbb{R}^{n\times p}$

$AB \in \mathbb{R}^{m\times p}$ $\quad (AB)^T \in \mathbb{R}^{p\times m}$

$B^T \in \mathbb{R}^{p\times n}$ $\quad A^T \in \mathbb{R}^{n\times m}$ $\quad B^T A^T \in \mathbb{R}^{p\times m}$

Matrix functions: matrices where the entries are functions

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & & & \\ a_{m1}(t) & a_{m2}(t) & \cdots & a_{mn}(t) \end{pmatrix}$$

$$\frac{dA}{dt} = \begin{pmatrix} a_{11}'(t) & a_{12}'(t) & \cdots & a_{1n}'(t) \\ a_{21}'(t) & a_{22}'(t) & \cdots & a_{2n}'(t) \\ \vdots & & & \\ a_{m1}'(t) & a_{m2}'(t) & \cdots & a_{mn}'(t) \end{pmatrix}$$

A $m\times n$ $\quad B$ $n\times p$

$$\Longrightarrow \frac{d}{dt}(A\cdot B) = \frac{dA}{dt}\cdot B + A\cdot\frac{dB}{dt} \quad \text{(Product rule)}$$