§ 2.6 Systems of Differential Equations. A First Look

Coupled differential equations

$$\begin{cases} \dfrac{dx}{dt} = 2x - xy \\ \dfrac{dy}{dt} = -3y + \frac{1}{2}xy \end{cases}$$

Decoupled systems

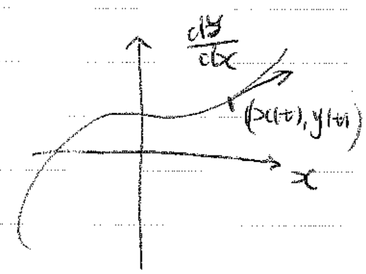
$$\begin{cases} \dfrac{dx}{dt} = 2x \\ \dfrac{dy}{dt} = -3y \end{cases}$$

Def'n: A solution of a system of two differential equations is a pair of functions $x(t)$ and $y(t)$ that simultaneously satisfies both equations.

e.g. $\begin{cases} x(t) = C_1 e^{2t} \\ y(t) = C_2 e^{-3t} \end{cases}$

$$\begin{cases} \dfrac{dx}{dt} = P(x,y) \\ \dfrac{dy}{dt} = Q(x,y) \end{cases}$$

autonomous two-dimensional first-order system



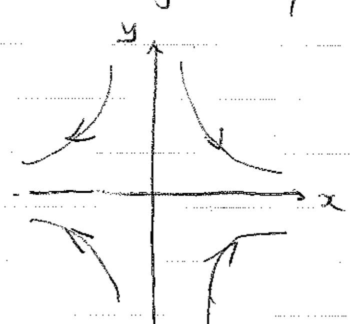
$(x(t), y(t))$ represents a parametric curve

$(x(0), y(0))$ is the initial condition

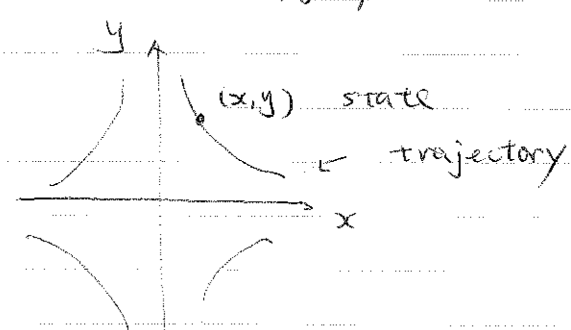
tangent vector $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$ if $\dfrac{dx}{dt} \neq 0$

Phase plane for a DE system in two variables

- phase plane : xy - plane
- vector field : collection of tangent vectors
- trajectory : parametric curve $(x(t), y(t))$



vector field



solution curves (phase portrait)

Equilibria, Nullclines

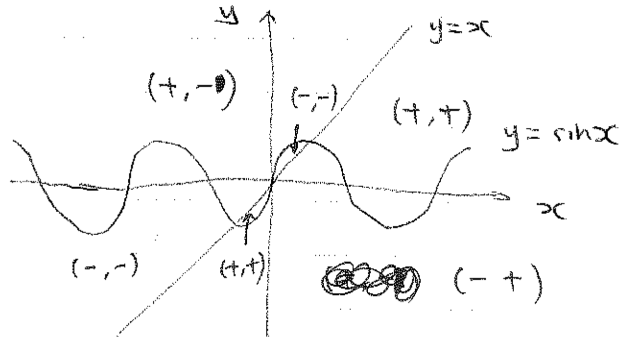
A v nullcline is an isocline of vertical slopes, where $\dfrac{dx}{dt} = 0$

A h nullcline is an isocline of horizontal slopes, $\dfrac{dy}{dt} = 0$

Equilibrium point: ① $\dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} = 0$

$\begin{cases} \text{stable: attract nearby solutions} \\ \text{unstable: repels nearby solutions in at least one direction} \end{cases}$

Ex: $\begin{cases} \dfrac{dx}{dt} = y - \sin x \\ \dfrac{dy}{dt} = x - y \end{cases}$



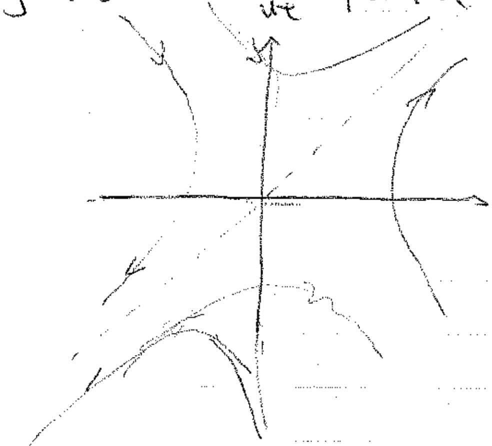
v nullcline: $y = \sin x$

h nullcline: $y = x$

equilibra: $x = \sin x$

$y - \sin x > 0$ $\quad \frac{dx}{dt}$ positive $\qquad y - \sin x < 0$ negtive

$x - y > 0$ $\quad \frac{dy}{dt}$ positive $\qquad x - y < 0$ negtive



The Lotka - Volterra Predator - Prey Model

Two species: $\begin{cases} \text{predator (wolves or foxes)} \quad \text{sharks} \\ \text{prey (sheep, or rabbits)} \quad \text{food fish} \end{cases}$

the same environment (woodland)

Assumptions:

① In the absence of predator, the prey population follows

$$\frac{dR}{dt} = a_R R. \quad a_R > 0$$

(Enough food for prey)

② In the absence of prey, the predator population follows

$$\frac{dF}{dt} = -a_F F, \quad a_F > 0$$

③ When both are present, the number of interactions is proportional to the product of the population sizes

The rate of change in the prey population is $-C_R RF$
The rate of change in the predator population is $+C_F RF$

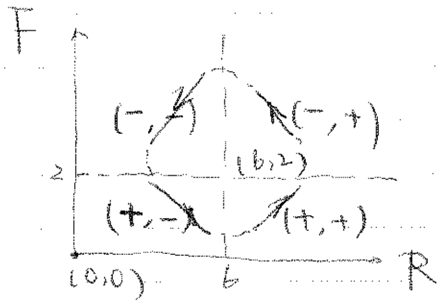
$C_F, C_R > 0$

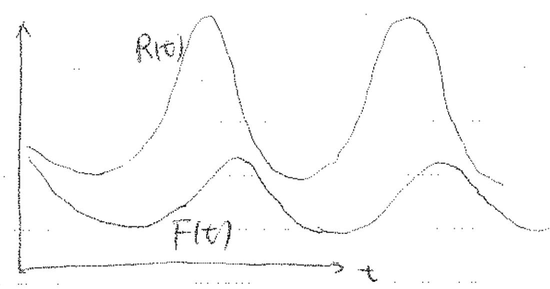
Based on the above assumptions, we have

$$\begin{cases} \frac{dR}{dt} = a_R R - C_R RF \\ \frac{dF}{dt} = -a_F F + C_F RF \end{cases}$$

a_R, a_F, C_R, C_F obtained by system identification

Ex:
$$\begin{cases} \frac{dR}{dt} = 2R - RF = R(2-F) \\ \frac{dF}{dt} = -3F + \frac{1}{2}RF = F(-3+\frac{1}{2}R) \end{cases}$$





The Competition Model

Two species: sheep, rabbits limited resources

Assumption:

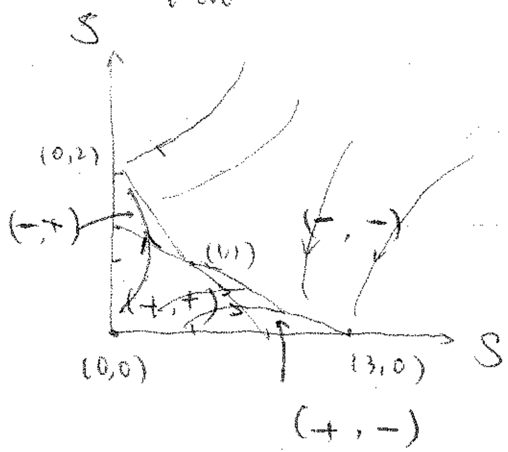
① Each species in the absence of the other follows the logistic law
$$\frac{dR}{dt} = a_R R\left(1 - \frac{R}{L_R}\right) \qquad L_R\text{: carrying capacity}$$
$$\frac{dS}{dt} = a_S S\left(1 - \frac{S}{L_S}\right) \qquad L_S\text{: carrying capacity}$$

② When grazing together, each species has a negative effect on the other due to limited resource.

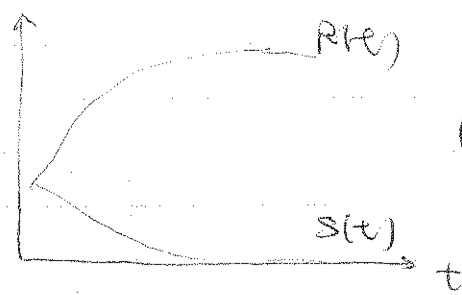
Rabbits: $-C_R RS$
Sheep: $-C_S RS$

$$\begin{cases} \frac{dR}{dt} = R(a_R - b_R R - C_R S) \\ \frac{dS}{dt} = S(a_S - b_S S - C_S R) \end{cases}$$

Ex:
$$\begin{cases} \dfrac{dR}{dt} = R(3 - R - 2S) \\ \dfrac{dS}{dt} = S(2 - S - R) \end{cases}$$



principle of competitive principle



(or switch)