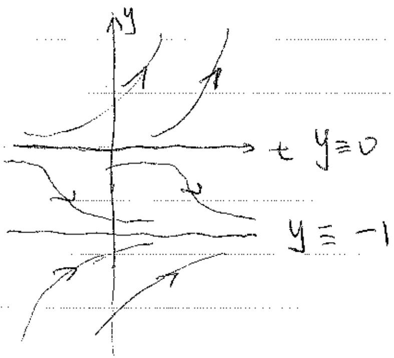
§2.5 Nonlinear Models: Logistic Equation

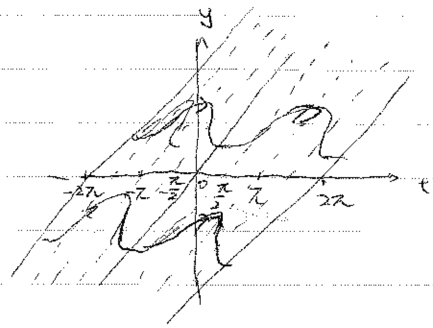
Nonlinear DEs:

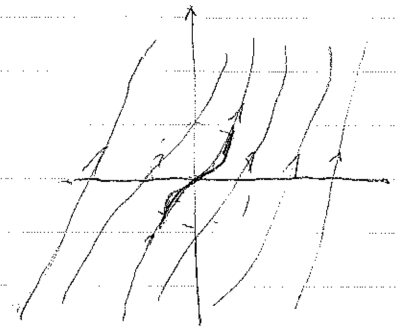
$$\frac{dy}{dt} = y(y+1) \qquad \frac{dy}{dt} = \sin(y-t) \qquad \frac{dy}{dt} = t^2+y^2$$

Quantitatively: $y(t) = \dfrac{Ce^t}{1-Ce^t}$, $C \in \mathbb{R}$ $\qquad \times \qquad \times$

Qualitatively:







Equilibria $\quad y \equiv 0$ unstable $\qquad \times \qquad \times$

$\qquad\qquad y \equiv -1$ stable

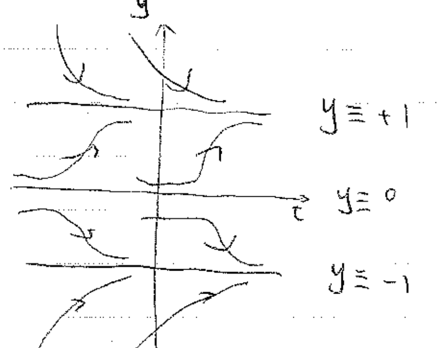
Autonomous Differential Equation:

A differential equation is autonomous if $\frac{dy}{dt} = f(y)$, that is, the right hand side does not explicitly include t.

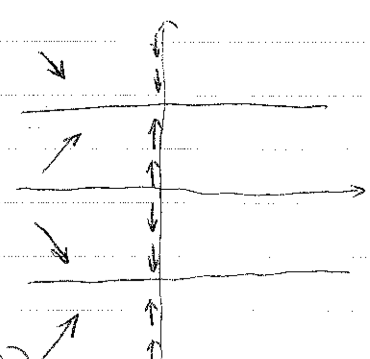
Phase line (similar direction fields). equilibria

- stable $\leftrightarrow$ sink
- unstable $\leftrightarrow$ source
- semistable $\leftrightarrow$ node

Ex: $y' = y(2-2y)(2+2y)$



direction fields



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Equilibra:

$y = -1, \ 0, \ 1$

stable unstable stable

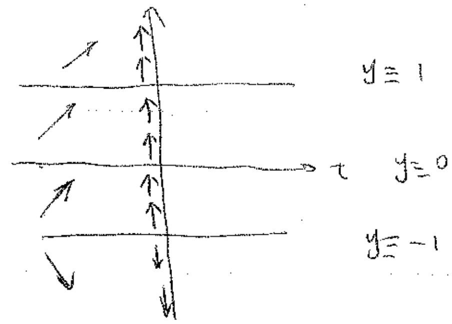
sink source sink

Ex: $y' = y^2(y-1)^2(y+1)$

Equilibria: $y = -1, \quad 0, \quad 1$

source node node

unstable semistable



$\frac{dy}{dt} = ky, \quad k > 0$

If more issues are considered in modeling, k should be generalized to a variable growth rate $k(y)$

If $k(y) = r - ay, \quad r > 0, \quad a > 0$

$\frac{dy}{dt} = (r - ay)y$ is called the logistic equation

Letting $L = \frac{r}{a}$, we have $\frac{dy}{dt} = r\left(1 - \frac{y}{L}\right)y$

$r > 0$ is called the initial (or intrinsic) growth rate

$L > 0$ is called the carrying capacity.

The explicit solution can be obtained by seperation of variables, or solving Bernouilli's equation.

$$y(t) = \frac{Ce^{rt}}{1 - \frac{C}{L}e^{rt}}$$

$$\begin{cases} \frac{dy}{dt} = r\left(1 - \frac{y}{L}\right)y \\ y(0) = y_0 \end{cases} \qquad y(t) = \frac{L}{1 + \left(\frac{L}{y_0} - 1\right)e^{-rt}}$$

Populations with Minimum Thresholds

$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)y \leftarrow$ threshold equation

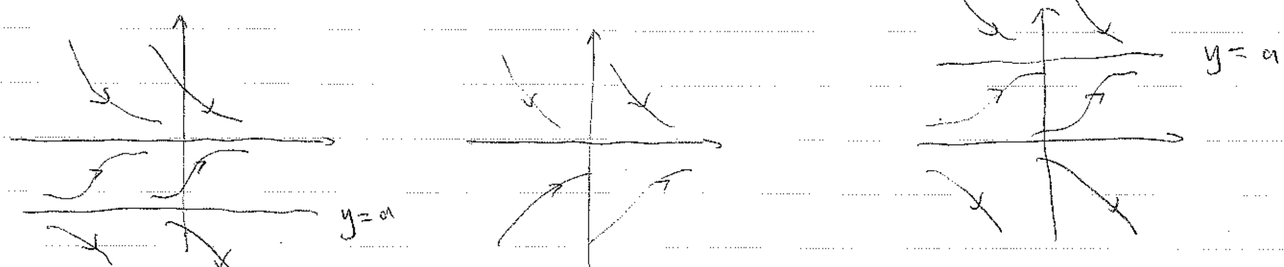
$r > 0$ is the initial growth rate, $T > 0$ is the threshold level

$$y(t) = \frac{T}{1 + \left(\frac{T}{y_0} - 1\right)e^{rt}}$$

If $y_0 < T$, then $y(t) \to 0$ as $t \to +\infty$

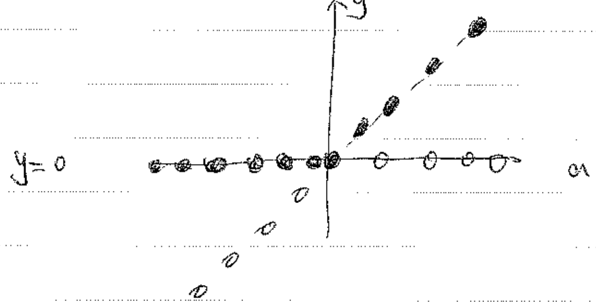
The species will become extinct.

Bifurcation: $y' = y(a-y)$ $\quad a \in \mathbb{R}$ $\quad$ Equilibra: $y \equiv 0$, $y \equiv a$



$a < 0$ $\qquad$ $a = 0$ $\qquad$ $a > 0$

$y = a$ source (unstable) $\qquad$ $y = a$ sink (stable) $\qquad$ $y = a$ sink (stable)



Bifurcation Diagram