§2.3 Growth and Decay Phenomena

Basic linear differential equation

$$\frac{dy}{dt} = ky$$

$$\begin{cases} k>0, & k \text{ is called the growth constant, or rate of growth} \\ k<0, & k \text{ is called the decay constant, or rate of decline} \end{cases}$$

$$\begin{cases} k>0, & \text{growth equation, e.g. population growth} \\ k<0, & \text{decay equation} \end{cases}$$

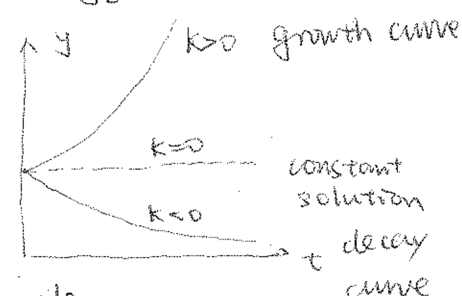
$$\frac{dy}{dt} = ky \rightarrow \frac{dy}{y} = k\,dt \rightarrow \ln|y| = kt + C$$

$$\rightarrow |y| = e^{C} e^{kt} \qquad e^{C} > 0, \ C \in \mathbb{R}$$

$$\rightarrow y(t) = \tilde{C} e^{kt}, \qquad \tilde{C} \in \mathbb{R}$$

For the IVP $\begin{cases} \frac{dy}{dt} = ky \\ y(0) = y_0 \end{cases}$ $\quad y(t) = y_0 e^{kt}$

$$\begin{cases} \text{growth curve} & \text{if } k>0 \\ \text{decay curve} & \text{if } k<0 \end{cases}$$



Ex: Radioactive Decay $\begin{cases} 6 \text{ proton} \\ 8 \text{ neutron} \end{cases}$

archaeology

The quantity Q of C-14 in a charcoal sample satisfies the decay equation $Q' = kQ$

The half-life of C-14 is approximately 5600 years

$$\begin{cases} Q' = kQ \\ Q(0) = Q_0 \end{cases} \rightarrow Q = Q_0 e^{kt}$$

$$Q(5600) = \frac{1}{2} Q_0 \rightarrow e^{5600t} = 0.5$$

$$\rightarrow t = \frac{\ln 0.5}{5600} \approx -0.00012378$$

If the amount of C-14 remaining in a charcoal sample was 15%, what is the age of the charcoal?

$$0.15 = \frac{Q(t)}{Q_0} = e^{-0.00012378\,t} \rightarrow t = -(5600 \ln 0.15)/\ln 2$$

$$\approx 15327 \text{ (years)}$$

Ex Compound Interest

Original amount A_0 Interest Rate r

Number of years	0	1	2	...	$N=t$
Future value	A_0	$A_0(1+r)$	$A_0(1+r)^2$		$A_0(1+r)^t$

Usually the bank pays interest n times per year.

$$A(t) = A_0\left(1+\frac{r}{n}\right)^{nt}$$

e.g. $A_0 = 100\$$ $r=0.06$ $n=12$ $t=2$ years

$$A(2) = \$100\left(1+\frac{0.06}{12}\right)^{24} = \$112.72$$

Suppose the bank makes its payments more and more often daily, hourly, every minute, every second, ... continuously

$$A(t) = \lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt} = A_0 e^{rt}$$

by using $\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{\frac{n}{r}} = e$

Remark: Continuous Compounding of Interest

If an initial amount of A_0 dollars is deposited at an annual interest rate of r, compounded continuously, the future value $A(t)$ of the deposit at time t satisfies the initial-value problem

$$\frac{dA}{dt} = rA \qquad A(0)=A_0$$

$$\to \quad A(t) = A_0 e^{rt}$$

Ex: Canarsie Indians sold the island of Manhattan at the price of \$24 in 1626. In 2000

$t = 2000-1626 = 374$

$A(374) = 24\, e^{0.08\cdot 374} = \$236,756,625,000,000$

Time is money!

Ex: If steady contributions are added

$$\begin{cases} \frac{dA}{dt} = rA + a \\ A(0) = A_0 \end{cases} \to \quad A(t) = A_0 e^{rt} + \frac{a}{r}(e^{rt}-1)$$

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