§2.2 Solving the first-order linear differential equation

$\begin{cases} \text{Euler-Lagrange Two-Stage Method} \\ \text{Integrating Factor Method} \end{cases}$

Consider $y' + p(t)y = f(t)$

Step I: all solutions to the homogeneous problem

$$y' + p(t)y = 0$$

$$\rightarrow y_h = c e^{-\int p(t)dt} \qquad c \text{ constant}$$

Step II: a solution to the inhomogeneous problem

Lagrange: variation of parameters

Assume $y_p = v(t) e^{-\int p(t)dt}$, since $y_p' + p(t) y_p = f(t)$

we have

$$v'(t) e^{-\int p(t)dt} + v(t) e^{-\int p(t)dt}(-p(t)) + p(t)v(t)e^{-\int p(t)dt} = f(t)$$

$$\rightarrow v'(t) e^{-\int p(t)dt} = f(t)$$

$$\rightarrow v'(t) = f(t) e^{\int p(t)dt}$$

$$\rightarrow v(t) = \int f(t) e^{\int p(t)dt} dt$$ (omit constant, only one solution is needed)

$$y_p = e^{-\int p(t)dt} \int f(t) e^{\int p(t)dt} dt$$

Step III: $y(t) = y_h(t) + y_p(t)$

$$= c e^{-\int p(t)dt} + e^{-\int p(t)dt} \int f(t) e^{\int p(t)dt} dt$$

Ex: $\begin{cases} y' + y = \cos t \\ y(0) = 1 \end{cases}$

Step I: $y' + y = 0 \rightarrow y_h = c e^{-t}$

Step II: $y_p = v(t) e^{-t}$ $\quad y_p$ satisfies $y_p' + y_p = \cos t$

Plug in $y_p$, we get

$$v'(t) e^{-t} + v(t) e^{-t}(-1) + v(t) e^{-t} = \cos t$$

$$\rightarrow v'(t) = e^t \cos t$$

$$\rightarrow v(t) = \int e^t \cos t \, dt$$

Let $F(t) = \int_0^t e^s \cos s \, ds$

$$F(t) = \int_0^t e^s \, d(\sin s) = e^s \sin s \Big|_0^t - \int_0^t \sin s \, e^s ds$$

$$= e^t \sin t + \int_0^t e^s \, d(\cos s) \quad (1)$$

$$= e^t \sin t + e^s \cos(s)\Big|_0^t - \int_0^t \cos s \, e^s \, ds$$

$$= e^t \sin t + e^t \cos t - I - F(t)$$

$$\longrightarrow \quad F(t) = \frac{1}{2}\left(e^t \sin t + e^t \cos t - I\right) = \frac{1}{2} e^t (\sin t + \cos t) - \frac{1}{2}$$

$$y_p = \frac{1}{2}(\sin t + \cos t) - \frac{1}{2} e^{-t}$$

$$y = y_h + y_p = \left(\frac{1}{2} + c\right) e^{-t} + \frac{1}{2}(\sin t + \cos t)$$

Since $y(0) = 1$. $\quad C = 1$

$$y(t) = \frac{1}{2} e^{-t} + \frac{1}{2}(\sin t + \cos t)$$

Integrating factor method:

$$y' + p(t) y = f(t)$$

Integrating factor $\mu(t) = e^{\int p(t)\, dt}$, then

$$\mu(t)\left(y' + p(t) y\right) = \mu(t) f(t)$$

$$\longrightarrow \quad e^{\int p(t) dt} y' + e^{\int p(t) dt} p(t) y = e^{\int p(t) dt} f(t)$$

$$\longrightarrow \quad \frac{d}{dt}\left(e^{\int p(t) dt} y\right) = e^{\int p(t) dt} f(t)$$

$$\longrightarrow \quad e^{\int p(t) dt} y = c + \int e^{\int p(t) dt} f(t)\, dt$$

$$\longrightarrow \quad y(t) = \underbrace{c e^{-\int p(t) dt}}_{y_h} + \underbrace{e^{-\int p(t) dt} \int e^{\int p(t) dt} f(t)\, dt}_{y_p}$$

Ex: $\begin{cases} y' + y = 1 \\ y(0) = 1 \end{cases}$

Integrating factor $\quad e^{\int 1\, dt} = e^t$

$$e^t y' + e^t y = e^t \longrightarrow \frac{d}{dt}(e^t y) = e^t$$

$$e^t y = e^t + C \longrightarrow y(t) = \underbrace{c e^{-t}}_{y_h} + \underbrace{1}_{y_p}$$

$$y(0) = 1 \implies c = 0$$

$$\longrightarrow \quad y(t) \equiv 1$$

Ex: $y' + \frac{1}{t} y = \frac{1}{t^2}$ $\quad t>0$

Integrating factor $e^{\int \frac{1}{t} dt} = e^{\ln t} = t$

$ty' + y = \frac{1}{t} \rightarrow (ty)' = \frac{1}{t}$

$ty = \int \frac{1}{t} dt = \ln t + C$

$\rightarrow y(t) = \frac{C}{t} + \frac{\ln t}{t}$

Ex: $y' + y = \frac{1}{1+e^{2t}}$

$\frac{d}{dt}(e^t y) = \frac{e^t}{1+e^{2t}} \rightarrow e^t y = \int \frac{e^t}{1+e^{2t}} dt + C$

let $u = e^t$ $\quad \int \frac{e^t}{1+e^{2t}} dt = \int \frac{du}{1+u^2} = \arctan u$

$= \tan^{-1} u = \tan^{-1} e^t$

$e^t y = \tan^{-1}(e^t) + C$

$\rightarrow y(t) = e^{-t} \tan^{-1}(e^t) + Ce^{-t}$

Ex: $y' - \frac{2}{t} y = t^3 \cos t$

Integrating factor $e^{-\int \frac{2}{t} dt} = \frac{1}{t^2}$

$\frac{d}{dt}\left(\frac{1}{t^2} y\right) = t \cos t$

$\frac{1}{t^2} y = \int t \cos t \, dt + C$

$= t \sin t + \cos t + C$

$\rightarrow y(t) = t^3 \sin t + t^2 \cos t + Ct^2$

Ex. Bernoulli's Equation

$y' + p(t) y = g(t) y^\alpha \qquad \alpha \in \mathbb{R} \quad \alpha \neq 0, \alpha \neq 1$

$\frac{y'}{y^\alpha} + p(t) y^{1-\alpha} = g(t)$

Define $u(t) = y^{1-\alpha}(t) \qquad \frac{du}{dt} = 1-\alpha \frac{y'}{y^\alpha}$

$\rightarrow \frac{1}{1-\alpha} u' + p(t) u = g(t)$

can be solved by Integrating Factor Method e.g., $y' + ty = ty^2 \qquad y(0) = 1$

Define $u(t) = y^{-1}$ then

$$-u' + tu = t$$

$$\longrightarrow \quad u' - tu = -t$$

$$\longrightarrow \quad \frac{d}{dt}\left(e^{-\frac{t^2}{2}} u\right) = -t\, e^{-\frac{t^2}{2}}$$

$$\longrightarrow \quad e^{-\frac{t^2}{2}} u = e^{-\frac{t^2}{2}} + C$$

$$\longrightarrow \quad \frac{1}{y(t)} = 1 + C e^{\frac{t^2}{2}}$$

$y(0) = 1 \implies C = 0 \implies y(t) \equiv 1$

If $y(0) = 2 \implies C = -\frac{1}{2} \implies y(t) = \dfrac{1}{1 - \frac{1}{2} e^{\frac{t^2}{2}}}$

Ex: Ricatti Equation

$$y' = p(t) + q(t)\, y + r(t)\, y^2$$

e.g. $y' = -1 + 2y - y^2$

$y_1 \equiv 1$ is an equilibrium solution of the equation

let $z = y - y_1 = y - 1$, then $y = z + 1$, and

$z' = -z^2 \quad \leftarrow$ Bernoulli's equation

$z = \dfrac{1}{t+C} \quad \longrightarrow \quad y(t) = 1 + \dfrac{1}{t+C}$

If $y_1$ satisfies ~~the~~ Ricatti Equation, i.e.,

$$y_1' = p(t) + q(t)\,y_1 + r(t)\,y_1^2$$

then $u = y - y_1$ satisfies

$$u' = q(t)u + r(t)\,(u + 2y_1)\,u$$

$$\rightarrow \quad u' = \left(q(t) + 2y_1(t)\,r(t)\right)u + r(t)\,u^2$$

Bernoulli's equation

Ex: $\dfrac{dy}{dx} = \dfrac{1}{e^{-y} - x}$

$$\frac{dx}{dy} = e^{-y} - x \quad \rightarrow \quad \frac{dx}{dy} + x = e^{-y}$$

Integrating Factor $e^{\int 1\, dy} = e^{y}$

$$\frac{d}{dy}(e^{y}x) = 1 \quad \rightarrow \quad e^{y}x = y + C$$

$$\rightarrow \quad x = e^{-y}y + Ce^{-y}$$

Ex: IVP

$$y' - \frac{4}{x}y = \begin{cases} -2x - \dfrac{4}{x}, & x \in [1,2) \\ x^2, & x \in (2,4] \end{cases}$$

$$y(1) = 1$$

Integrating Factor $e^{\int -\frac{4}{x}dx} = \dfrac{1}{x^4}$

$$\frac{d}{dx}\left(\frac{y}{x^4}\right) = \begin{cases} -\dfrac{2}{x^3} - \dfrac{4}{x^5} & x \in [1,2) \\ \dfrac{1}{x^2} & x \in (2,4] \end{cases}$$

$$y(x) = \begin{cases} x^2 + 1 + C_1x^4 & x \in [1,2) \\ -x^3 + C_2x^4 & x \in (2,4] \end{cases}$$

$$y(1) = 1 \quad \rightarrow \quad C_1 = -1$$

$$\lim_{x\to 2^-} y(x) = \lim_{x \to 2^+} y(x) \quad \longrightarrow \quad C_2 = -\frac{3}{16}$$

$$y(x) = \begin{cases} -x^4 + x^2 + 1 & x \in [1,2) \\ -\frac{3}{16}x^4 - x^3 & x \in (2,4] \end{cases}$$

$$\lim_{x\to 2^-} y'(x) = -20 \quad \lim_{x\to 2^+} y'(x) = -18$$