§ 2.1 Linear Equations: The Nature of Their Solutions

Def: An (algebraic) equation is called linear if it is of the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = C$$

where $a_1, a_2, \cdots, a_n$ and C are constants.

If $C = 0$, the equation is said to be homogeneous.

$$V = \left\{ \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \;\middle|\; \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \text{ and } a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0 \right\}$$

is a vector (sub)space over $\mathbb{R}$

Def: A differential equation of order n is called linear if it is of the form

$$a_0(t)\, y^{(n)} + a_1(t)\, y^{(n-1)} + \cdots + a_{n-1}(t)\, y' + a_n(t)\, y = f(t)$$

If $f(t) \equiv 0$, the equation is said to be homogeneous.

$$V = \left\{ y(t) \;\middle|\; y(t) \in C^n(\mathbb{R}) \text{ and } a_0(t)\, y^{(n)} + \cdots + a_n(t)\, y = 0 \right\}$$

is a vector (sub)space over $\mathbb{R}$

Ex: $y'' + \sin(t)\, y' - \log(1+t^2)\, y = e^t$ is linear

Introduce the operator T:

① For linear algebraic equation, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$$T(\vec{x}) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

② For linear differential equation, $y \in C^n(\mathbb{R})$

$$T(y) = a_0(t)\, y^{(n)} + a_1(t)\, y^{(n-1)} + \cdots + a_{n-1}(t)\, y' + a_n(t)\, y$$

Def: Given two vector spaces V, W, an operator (map) $T: V \to W$ is said to be linear if:

(1) $T(\vec{v_1} + \vec{v_2}) = T(\vec{v_1}) + T(\vec{v_2}) \quad \forall\, \vec{v_1}, \vec{v_2} \in V$

(2) $T(c\,\vec{v_1}) = c\, T(\vec{v_1}) \quad \forall\, v_1 \in V,\ \forall\, c$ scalar

Ex: ① $T: \mathbb{R}^n \longrightarrow \mathbb{R}$

$\vec{x} \longmapsto a_1x_1 + a_2x_2 + \cdots + a_nx_n = T(\vec{x})$, a_i scalars

T is a linear operator.

① $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$? $\forall\ \vec{x}, \vec{y} \in \mathbb{R}^n$

$$T(\vec{x}+\vec{y}) = T\left(\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{pmatrix}\right) = a_1(x_1+y_1) + a_2(x_2+y_2) + \cdots + a_n(x_n+y_n)$$

$$= (a_1x_1 + a_2x_2 + \cdots + a_nx_n) + (a_1y_1 + a_2y_2 + \cdots + a_ny_n)$$

$$= T(\vec{x}) + T(\vec{y})$$

② $T(c\vec{x}) = c\,T(\vec{x})$? $\forall\ \vec{x} \in \mathbb{R}^n$, c scalar

$$T(c\vec{x}) = T\left(\begin{pmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{pmatrix}\right) = a_1(cx_1) + a_2(cx_2) + \cdots + a_n(cx_n)$$

$$= c(a_1x_1 + a_2x_2 + \cdots + a_nx_n) = c\,T(\vec{x})$$

② $C^2(\mathbb{R})$, and $C(\mathbb{R})$ are vector spaces

Assume $p(t)$, $q(t)$ are continuous functions

$T: C^2(\mathbb{R}) \longrightarrow C(\mathbb{R})$

$y \longmapsto y'' + p(t)y' + q(t)y = T(y)$

T is a linear operator

① $T(y_1+y_2) = T(y_1) + T(y_2)$? $\forall\ y_1, y_2 \in C^2(\mathbb{R})$

$$T(y_1+y_2) = (y_1+y_2)'' + p(t)(y_1+y_2)' + q(t)(y_1+y_2)$$

$$= y_1'' + y_2'' + p(t)(y_1' + y_2') + q(t)(y_1+y_2)$$

$$= \left(y_1'' + p(t)y_1' + q(t)y_1\right) + \left(y_2'' + p(t)y_2' + q(t)y_2\right)$$

$$= T(y_1) + T(y_2)$$

② $T(cy_1) = c\,T(y_1)$? $\forall\ y \in C^2(\mathbb{R})$ $\forall\ c$ scalar

$$T(cy_1) = (cy_1)'' + p(t)(cy_1)' + q(t)(cy_1)$$

$$= cy_1'' + p(t)\,c\,y_1' + q(t)\,cy_1$$

$$= c\,(y_1'' + p(t)y_1' + q(t)y_1)$$

$$= c\,T(y_1)$$

Superposition Principle for linear homogeneous equations

Let $\vec{u}_1$ and $\vec{u}_2$ be any solutions of the homogeneous linear equation $T(\vec{u}) = 0$, then

(1) $\vec{u_1} + \vec{u_2}$ is also a solution
(2) $c\vec{u_1}$ is also a solution for any constant C
e.g. $T(\vec{x}) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$
$T(y) = y'' + p(t) y' + q(t) y = 0$

Nonhomogeneous Principle
Let $\vec{u_p}$ be any solution (called a particular solution) to a linear nonhomogeneous equation
$L(\vec{u}) = C$ (algebraic) or $L(\vec{u}) = f(t)$ (differential)
Then $\vec{u} = \vec{u_h} + \vec{u_p}$ is also a solution, where
$\vec{u_h}$ is a solution to the associated homogeneous equation
$L(\vec{u}) = 0$
Furthermore, every solution of the nonhomogeneous equation must be of the form $\vec{u} = \vec{u_h} + \vec{u_p}$

Proof: ① $L(\vec{u}) = L(\vec{u_h} + \vec{u_p}) = L(\vec{u_h}) + L(\vec{u_p})$
$= 0 + f(t) = f(t)$

② $\forall \vec{u}$ $L(\vec{u}) = f(t)$ and $L(\vec{u_p}) = f(t)$ then
$\vec{u_h} = \vec{u} - \vec{u_p}$
$L(\vec{u_h}) = L(\vec{u} - \vec{u_p}) = L(\vec{u}) + L((-1)\cdot\vec{u_p})$
$= L(\vec{u}) + (-1) L(\vec{u_p}) = f(t) - f(t) = 0$
$\vec{u_h}$ is a solution to the associated homogeneous equation

Procedure for solving nonhomogeneous equations
Step I: Find all $\vec{u_h}$ of $L(\vec{u}) = 0$
Step II: Find any $\vec{u_p}$ of $L(\vec{u}) = f(t)$
Step III: $\vec{u} = \vec{u_h} + \vec{u_p}$, to get all solutions of $L(\vec{u}) = f(t)$

Ex: $x_1 + x_2 - x_3 = 1$
$[\,1 \;\; 1 \;\; -1 \mid 1\,]$
Choose x_2, x_3 as parameters then $x_1 = -x_2 + x_3 + 1$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 + x_3 + 1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{\text{all solutions to the homogeneous equation}} + \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\text{particular solution}}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \begin{aligned} \mathbb{R}^3 &\to \mathbb{R} \\ L: \vec{x} &\to x_1 + x_2 - x_3 \end{aligned}$$

Step 1: $L(\vec{x}) = 0$, i.e., $x_1 + x_2 - x_3 = 0$

$\vec{x}_h = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ all solutions

Step II: $L(\vec{x}) = 1$, i.e., $x_1 + x_2 - x_3 = 1$

$x_2 = 0$, $x_3 = 0 \to x_1 = 1$

$\vec{x}_p = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ only one solution

Step III: $\vec{x} = \vec{x}_h + \vec{x}_p$ all solutions to $L(\vec{x}) = 1$