§1.5 Picard's Theorem: Theoretical Analysis

Mathematical Modeling from real world: DE

- Does the modeling have solutions?
  Existence of ODE # solutions ≥ 1
- How many solutions?
  $\begin{cases} \text{Uniqueness} & \# \text{solutions} = 1 \to \text{prediction} \\ \text{Multiplicity} & \# \text{solutions} > 1 \end{cases}$
- What are they like?
  Qualitative theory
- How can we represent them?
  Explicit solution or numerical approximations

Consider the IVP $\begin{cases} y' = f(t,y) & \leftarrow \text{Nonlinear DE} \\ y(t_0) = y_0 \end{cases}$

Picard's Existence and Uniqueness Theorem (local)

Assume $f(t,y)$ is continuous over the region

$$R = \{(t,y) \mid a<t<b, \ c<y<d\}$$

and $(t_0, y_0) \in R$. Then there exists a positive number h such that the IVP has a solution for t in the interval (t_0-h, t_0+h).

If, furthermore, $f_y(t,y)$ is also continuous in R, that solution is unique.

IVP is equivalent to the following integral equation (nonlinear)

$$y(t) = y(t_0) + \int_{t_0}^{t} f(s, y(s))\, ds$$

$$y_{m+1}(t) = y(t_0) + \int_{t_0}^{t} f(s, y_m(s))\, ds$$

To prove the existence, we need to control

$$|y_m(t) - y(t)| \to 0, \quad \text{as } m \to \infty$$

over some interval

Uniqueness is trivial compared with existence

Ex: $\begin{cases} y' = -y \\ y(0) = 1 \end{cases}$ IVP

Integral equation

$$y(t) = y(0) - \int_0^t y(s)\,ds$$

Successive approximation: $y_0(t) = 1$

$$y_{m+1}(t) = y(0) - \int_0^t y_m(s)\,ds \qquad m = 0, 1, 2, \cdots$$

$m=0$, $y_1(t) = y(0) - \int_0^t ds = 1 - t$

$m=1$, $y_2(t) = y(0) - \int_0^t y_1(s)\,ds = 1 - t + \frac{1}{2!}t^2$

$m=2$, $y_3(t) = y(0) - \int_0^t y_2(s)\,ds = 1 - t + \frac{1}{2!}t - \frac{1}{3!}t^3$

By induction, we get

$$y_k(t) = \sum_{m=0}^{k} \frac{1}{m!}(-t)^m$$

As $k \to \infty$, we know $\lim_{k\to\infty} y_k(t) = e^{-t}$

Speration of variables also gives us $y(t) = e^{-t}$,

which means $\lim_{k\to\infty} |y_k(t) - y(t)| = 0$

Ex: $f(x) = 8x^3 + 2x - 1$ over $[0, 1]$

$f(0) = -1 \quad f(1) = 9$

Theorem of calculus tells us there is at least one root of $f(x)$

$f'(x) = 24x^2 + 2 > 0$

Rolle's Theorem: Only one root over the interval.

$$x = \frac{1}{4x^2+1}$$

successive approximation: $x_{k+1} = \dfrac{1}{4x_k^2+1} \quad k = 0, 1, 2, \cdots$

$x_0 = 0 \rightarrow x_1 = 1 \rightarrow x_2 = 0.2 \rightarrow x_3 = 0.8621$

Ten thousand iterations give us $x_{10000} = 0.49$

The exact value is $x_{root} = 0.50$