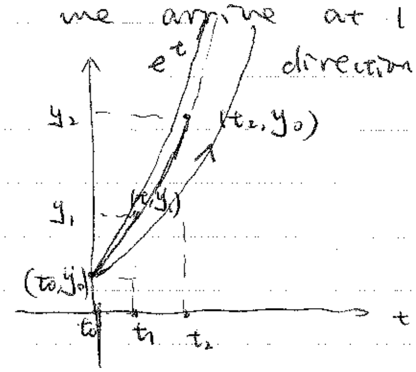
§ 1.4 Approximation Methods: Numerical Analysis

Euler's Method

$$\begin{cases} y'(t) = y \\ y(0) = 1 \end{cases} \qquad y(t) = e^t$$

- point $(0,1)$ slope (direction) $y' = f(t,y) = 1$
  We follow the line through $(0,1)$, move with slope 1, until we reach $t=1$. We arrive at $(1,2)$
- point $(1,2)$ slope $y' = 2$
  Moving another step, we arrive at $(2,4)$
- point $(2,4)$ slope $y' = 4$
  Again, we arrive at $(4,8)$ after another step



For the initial-value problem $\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$

Euler's Method works as:

$$\begin{cases} t_{n+1} = t_n + h = t_0 + (n+1)h \\ y_{n+1} = y_n + h\, f(t_n, y_n) \end{cases} \qquad h: \text{ step size}$$

Start from (t_0, y_0), $(t_1, y_1), (t_2, y_2), \cdots, (t_n, y_n)$ can be obtained iteratively. The piecewise-linear function connecting these points is the Euler approximation to the solution $y(t)$ of the IVP for $t_0 \le t \le t_K$

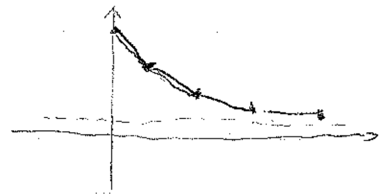
Ex.: $\begin{cases} y' = 1-2y \\ y(0) = 1 \end{cases}$ $\qquad h = 0.1$

$t_0 = 0 \qquad y_0 = 1$

$n=0 \qquad t_1 = t_0 + h = 0.1 \qquad y_1 = y_0 + h(1-2y_0) = 0.9$

$n=1 \qquad t_2 = 0.2 \qquad y_2 = y_1 + h(1-2y_1) = 0.82$

$n=2 \quad y_3 = y_2 + h(1-2y_2) = 0.756 \quad t_3 = 0.3$



compare with explicit solution

$$y(t) = \frac{1}{2}(e^{-2t}+1)$$

or direction field

<u>Note</u>: Remember that $\frac{dy}{dt}$ can be approximated by $\frac{y(t+h)-y(t)}{h}$, h small. We then obtain

$$y(t+h) - y(t) = h f(t,y)$$

Error estimates:

Roundoff error: from computers

Discretization error: from numerical methods

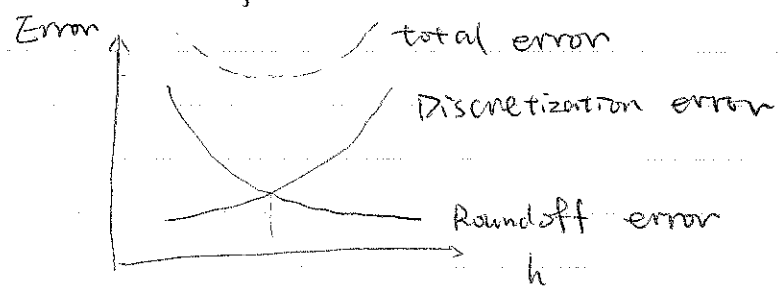
Local discretization error (Taylor series expansion)

$$|y_i - y(t_i)| \le C h^2$$

Global (accumulated) discretization error

$$|y_n - y(t_n)| \le C h \qquad n = 1, 2, \cdots, k \qquad \approx O(h)$$

The ORDER of Euler's Method is 1.



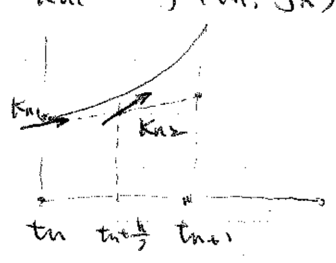
Runge-Kutta Method

① Second-Order Runge-Kutta Method (Midpoint Euler)

$$t_{n+1} = t_n + h$$

$$y_{n+1} = y_n + h K_{n2}$$

where $K_{n1} = f(t_n, y_n) \qquad K_{n2} = f(t_n + \frac{h}{2}, y_n + \frac{h}{2} K_{n1})$



Ex. $\begin{cases} y' = 1-2y \\ y(0) = 1 \end{cases}$ $h=0.1$ $t_0=0$ $y_0=1$

$n=0$ $t_1 = 0.1$ $k_{01} = -1$ $k_{02} = -0.9$ $y_1 = 0.91$ $|y_1 - y(t_1)| \approx 0.0006$

$y(t_1) = \frac{1}{2}(e^{-2\times 0.1}+1) \approx 0.9094$ $(0.909365376538\cdots)$

Roundoff error

$n=1$ $t_2 = 0.2$ $k_{11} = -0.82$ $k_{12} = -0.738$

$y_2 = 0.8362$ $y(t_2) \approx 0.8352$

$|y_2 - y(t_2)| \approx 0.001$ Discretization error

$\cdots$

In Euler's Method, $|y_1 - y(t_1)| = 0.01$

$|y_2 - y(t_2)| = 0.0152$

Second-order Runge-Kutta Method vs First-order Euler's Method

Fourth-Order Runge-Kutta Method

$t_{n+1} = t_n + h$

$y_{n+1} = y_n + \frac{h}{6}(k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4})$

where $k_{n1} = f(t_n, y_n)$

$k_{n2} = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_{n1})$

$k_{n3} = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_{n2})$

$k_{n4} = f(t_n + h, y_n + h k_{n3})$

Ex. $n=0$ $t=0.1$

$k_{n1} = -1$ $k_{n2} = -0.9$ $k_{n3} = -0.91$ $k_{n4} = -0.818$

$y_1 = 1 - 0.090633\cdots = 0.9093666\cdots$ circulating decimal

≈ 0.9094

$y(t_1) = 0.909365376538\cdots \approx 0.9094$

$|y_1 - y(t_1)| = 0$

or $|y_1 - y(t_1)| = 0.0000012\cdots$

Other methods: $\begin{cases} \text{Multistep methods} \\ \text{Variable step size methods} \\ \text{Symplectic methods} \end{cases}$