§1.3 Seperation of Variables: Quantative Analysis

Ex: $y' = y$ $\quad y(0) = 1$

equilibrium solution $y(t) \equiv 0$ does NOT satisfy the IVP

Now Assume $y \neq 0 \rightarrow \frac{1}{y(t)} \frac{d y(t)}{dt} = 1$

Integration on t over $[0, t]$

$$\int_0^t \frac{1}{y(s)} \frac{d y(s)}{ds} \, ds = \int_0^t 1 \, ds = t$$

Change of variables. $\quad Z = y(s)$

$\quad dZ = y'(s)\, ds$

$$\Rightarrow \int_0^t \frac{1}{Z} \, dZ = t \qquad \ln Z \Big|_{Z(0)=y(0)}^{Z(t)=y(t)} = t$$

$$\Rightarrow \ln y(t) - \ln y(0) = t$$

$$y(t) = e^{\ln y(0) + t} = e^{\ln y(0)} e^{t} = y(0) e^{t} = e^{t}$$

Seperation of variables for $y' = f(t)\, g(y)$

Ex: $y' = e^t y^2$ ✓ $\quad y' = e^{ty}$ ×

Step 1: Set $g(y) = 0$ and obtain equilibrium epuations

Step 2: Assume $g(y) \neq 0$. $\quad \frac{1}{g(y)} dy = f(t)\, dt$

Step 3. Integrate each side $\quad \int \frac{1}{g(y)} dy = \int f(t)\, dt + C$

$\Rightarrow \quad F(y(t)) = \int f(t)\, dt + C$

and solve for $y(t)$

Step 4: Use the initial ~~value~~ condition to evaluate C for IVP

Ex: $y' = y^2 \quad y(0) = 1$

$y(t) \equiv 0$ ×

/ $\quad \frac{y'}{y^2} = 1$ or $\frac{dy}{y^2} = dt \rightarrow \int \frac{dy}{y^2} = \int dt + C$

$$\Rightarrow -\frac{1}{y(t)} = t + C \Rightarrow y(t) = -\frac{1}{t+C}$$

$$y(0) = -\frac{1}{C} = 1 \Rightarrow C = -1$$

$y(t) = \frac{1}{1-t}$ only defined in $(-\infty, 1) \cup (1, +\infty)$

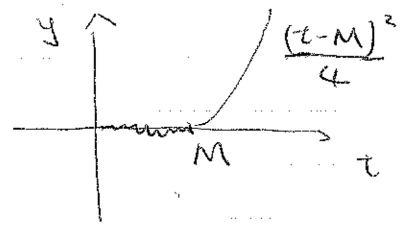
Note: $\int \frac{1}{z^p} dz = \int z^{-p} dz = \begin{cases} \frac{z^{-p+1}}{1-p} + C & p \neq 1 \\ \ln|z| + C & p = 1 \end{cases}$

Ex: Consider $\begin{cases} y' = \sqrt{|y|} & t \geq 0 \\ y(0) = 0 \end{cases}$ $y(t) \equiv 0$ is a solution

If $y(t) \neq 0$ at every t (assume $y \geq 0$)

$y' = \sqrt{y} \Rightarrow \frac{y'}{\sqrt{y}} = 1$ $\quad \int \frac{y'}{\sqrt{y}} dt = t + C$

$\rightarrow 2\sqrt{y} = t + C \rightarrow y = \frac{(t+C)^2}{4}$



then $z(t) = \begin{cases} 0 & 0 \leq t \leq M \\ \frac{(t-M)^2}{4} & t \geq M \end{cases}$

$z'(t) = \sqrt{z}$ if $0 \leq t < M$, or $t > M$

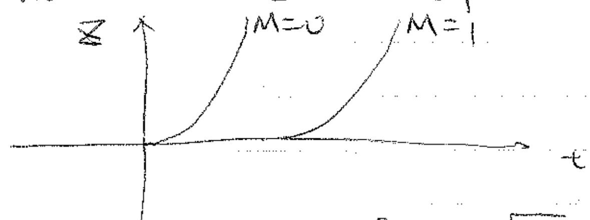
What is $\lim_{t \to M^-} z'(t) = ?$ $\quad 0$

$\lim_{t \to M^+} z'(t) = ?$ $\quad 0$ $\quad = \sqrt{z(M)}$

$\rightarrow z'(M) = \sqrt{z(M)}$

$\Rightarrow$ $z(t)$ is a solution of $\begin{cases} y'(t) = \sqrt{y} \\ y(0) = 0 \end{cases}$ for any $M \geq 0$

All these solutions are of the form:



The problem is that $f(y) = \sqrt{|y|}$ is not differentiable at $y = 0$, so Picard's thm does NOT apply.
f is continuous, but this example shows that continuity is not enough to ensure uniqueness.

Ex: $y' = y(1 - y^2)$

$\rightarrow \frac{y'}{y(1-y^2)} = 1 \rightarrow \int \frac{y'}{y(1-y^2)} dt = t + C$

Partial fractions

$$\frac{1}{y(1-y^2)} = \frac{1}{y(1-y)(1+y)} = \frac{A}{y} + \frac{B}{1-y} + \frac{C}{1+y}$$

$$\rightarrow \quad A(1-y^2) + By(1+y) + Cy(1-y) = 1$$

reduction of fractions to a common denominator

$$A + (B+C)y + (-A+B-C)y^2 = 1$$

$$A = 1 \qquad B+C=0 \qquad -A+B-C=0$$

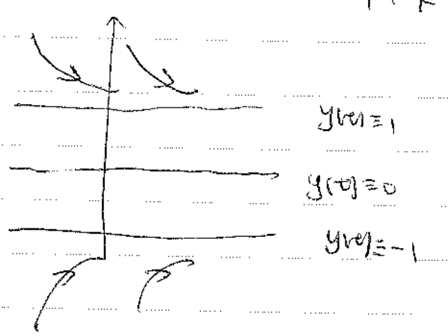
$$\rightarrow \quad A=1 \qquad B=\tfrac{1}{2} \qquad C=-\tfrac{1}{2}$$

$$\int \frac{dy}{y(1-y^2)} = \int \frac{dy}{y} + \frac{1}{2}\int \frac{dy}{1-y} - \frac{1}{2}\int \frac{dy}{1+y}$$

$$= \ln y - \tfrac{1}{2}\ln(1-y) - \tfrac{1}{2}\ln(1+y) \qquad 0<y<1$$

$$\Rightarrow \quad \ln \frac{y}{\sqrt{1-y^2}} = t + C \quad \Rightarrow \quad \frac{y^2}{1-y^2} = K^2 e^{2t} \qquad K = e^C$$

$$\Rightarrow \quad y^2(t) = \frac{K^2 e^{2t}}{1+K^2 e^{2t}} \qquad y(t) = \frac{Ke^t}{\sqrt{1+K^2e^{2t}}} \qquad 0<y<1$$



$y(t) > 1$ $\quad \lim_{t\to\infty} y(t) = 1$

$0 < y(t) < 1$ $\quad \lim_{t\to\infty} y(t) = 1$

$-1 < y(t) < 0$ $\quad \lim_{t\to\infty} y(t) = -1$

$y(t) < -1$ $\quad K<0$ $\quad \lim_{t\to\infty} y(t) = -1$

Def: $y' = f(t,y)$ is called autonomous if $f(t,y) = f(y)$ independent of t.
Otherwise, it is called nonautonomous