§1.2 Solutions and Direction Fields: Qualitative Analysis

Qualitative Analysis: direction fields
Quantitative Analysis: explicit solution, numerical solution

Def: To solve the differential equation $y' = f(t, y)$ means to find a function $z(t)$ such that

$$\frac{dz}{dt} = f(t, z(t)) \quad \text{for every } t \text{ in some interval}$$

Ex① $y' = 2t \rightarrow dy = 2t\,dt$

$y = t^2 + C$ is a family of solutions with parameter C

Such a family is called the general solution,
$y = t^2$ is called a particular solution

Ex② $\frac{dy}{dx} = y(x)$ $\qquad y(x) = 0$ $\qquad y(x) = e^x$ $\qquad y(x) = Ke^x$
K is a constant (parameter)

Initial-Value Problem (IVP)

$$\begin{cases} \frac{dy}{dt} = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

Its solution will pass through (t_0, y_0)

Ex: ① $y(0) = 0 \rightarrow y(t) = t^2$
② $y(0) = 1 \rightarrow y(x) = e^x$

Theorem: (Picard) If f is a C^1 function, then the initial value problem $\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$

has an unique solution

Direction field (slope field) of $y' = f(t, y)$
ty-plane, evaluate slope y' over the plane

Graphical Definition of solution.

A solution to a first-order DE is a function whose slope at each point is specified by the derivative.

Def (Equilibria):

A solution of DE is called an equilibrium solution if it does NOT change over time.

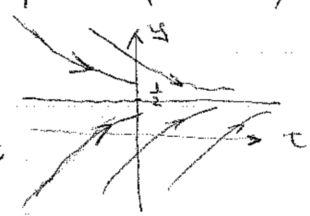
Note. For $y' = f(t, y)$, an equilibrium solution is a horizontal line $y(t) \equiv C$, which can be obtained by setting $y' = 0$

always equal, for every t

Ex: $y' = 1 - 2y$

$y' = 0 \Rightarrow y(t) \equiv \frac{1}{2}$ for every t

stable



Def: $y(t) \equiv C$ is called

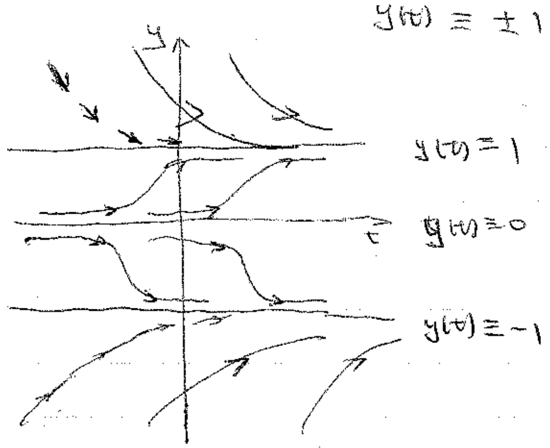
- stable if solutions near it tend toward it as $t \to \infty$
- unstable if solutions near it tend away from it as $t \to \infty$

Ex: $y' = y(1 - y^2)$

$y' = 0 \Rightarrow y(t) \equiv 0$

$y(t) \equiv \pm 1$

equilibrium solutions



$y(t) \equiv 1$ & $y(t) \equiv -1$ are stable

$y(t) \equiv 0$ is unstable

Def: An equilibrium point $y(t) \equiv C$ of $y' = f(t, y)$ is called asymptotically stable if $\lim_{t \to \infty} y(t) \equiv C$ for solutions starting close to C

This is also called a sink.

The Same Example: $y = \pm 1$ are asymptotically stable.

Def: Given an equilibrium solution $y(t) \equiv C$ of $y' = f(t,y)$, its basin of attraction is the set of points (initial values) such that the corresponding solution converges to C as $t \to +\infty$

Still the same example:

$y(t) \equiv +1 \rightarrow$ basin $= (0, +\infty)$

$y(t) \equiv -1 \rightarrow$ basin $= (-\infty, 0)$

$y(t) \equiv 0 \rightarrow$ basin $= \{0\}$