§ Fluids

For a fluid (gas / liquid), the basic quantities are

$\rho(\vec{x}, t)$: the mass density, which is a scalar

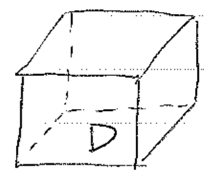
$\vec{V}(\vec{x}, t)$: the velocity of the fluid at point $\vec{x}$, time t

then the eulerian form of the fluid equations can be derived from conservation principles of mass, momentum, and energy

① Conservation of mass

Take any region D. The amount of mass within D at time t is $\iiint \rho(\vec{x}, t)\, d\vec{x}$. Fluid can exit the region only through the D boundary.

$$\frac{d}{dt}\iiint_D \rho(\vec{x}, t)\, d\vec{x} = -\iint_{\partial D} \rho \vec{v}\cdot\vec{n}\, dS$$



where $\rho\vec{v}\cdot\vec{n}$ is the rate of exiting in the unit outward normal direction $\vec{n}$ at point $\vec{x}$. The minus sign indicates that the mass within D is decreasing if the fluid is escaping from D.

By Leibniz Integral Rule, we get

$$\iiint_D \frac{\partial}{\partial t}\rho(\vec{x}, t)\, d\vec{x} = -\iint_{\partial D} \rho\vec{v}\cdot\vec{n}\, dS = -\iiint_D \nabla\cdot(\rho\vec{v})\, d\vec{x}$$

The second equality is obtained by applying the divergence thm.

Since D is arbitrary, applying the second vanishing thm, we have

$$\frac{\partial}{\partial t}\rho = -\nabla\cdot(\rho\vec{v}) \qquad ①$$

This is the equation of continuity

② Conservation of momentum

$$\frac{d}{dt}\iiint \rho v_i\, d\vec{x} = -\iint_{\partial D} \rho v_i \vec{V}\cdot\vec{n}\, dS - \iint_{\partial D} p n_i\, dS + \iiint_D \rho F_i\, d\vec{x}$$

where $p(\vec{x}, t)$ is the pressure and $F(\vec{x}, t)$ is the external forces. $i = 1, 2, 3$

The first term is the rate of change of momentum
The second term is the flux of momentum across the boundary
The third term is the net pressure at the boundary
The fouth term is the net external force.
The same argument as we did for the equation of continuity gives

$$\frac{\partial}{\partial t}(\rho v_i) = -\nabla\cdot(\rho v_i \vec{v}) - \frac{\partial P}{\partial x_i} + \rho F_i$$

Carrying out the derivatives, we get

$$\rho\left[\frac{\partial}{\partial t}v_i + \vec{v}\cdot\nabla v_i\right] + v_i\left[\frac{\partial}{\partial t}\rho + \nabla\cdot(\rho\vec{v})\right] = -\frac{\partial P}{\partial x_i} + \rho F_i$$

The second term in barckets vanishes due to ①. So we end up with the equation of motion

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = F - \frac{1}{\rho}\nabla P \qquad ②$$

We also need an equation for P. For example

$$P(x,t) = f(\rho(x,t)) \qquad \text{e.g. } P = c\rho^{\gamma}$$

which is the equation of state.

Conservation of Energy gives us an relationship between ρ, $\vec{v}$, P under certain assumptions.

If $F = \nu\nabla\vec{v}$, where $\nu > 0$ is the viscosity of the fluid

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = -\nabla\cdot(\rho\vec{v}) \\ \dfrac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = \nu\nabla\vec{v} - \dfrac{1}{\rho}\nabla P \end{cases}$$

is called the Navier - Stokes equation.
The well-posedness of NS equation is still open problem.
Turbulence is a consequence of the nonlinear character of the equation.

Remark: For incompressible fluid $\rho(\vec{x},t)$ is a constant, ① $\Rightarrow$ $\nabla\cdot\vec{v} = 0$ which means the divergence of velocity field is 0 everywhere. Physically, this is equivalent to saying that the local volume dilation rate is 0.