

§ 8.4 Approximations of Laplace's Equation

For Dirichlet's problem in a domain of irregular shape, it will be more convenient to compute numerically than to find the Green's function.

For $u_{xx} + u_{yy} = 0$, the natural approximation is that of centered differences,

$$\frac{u_{j+1,k} - 2u_{j,k} + u_{j-1,k}}{(\Delta x)^2} + \frac{u_{j,k+1} - 2u_{j,k} + u_{j,k-1}}{(\Delta y)^2} = 0$$

where $u_{j,k}$ is an approximation of $u(j\Delta x, k\Delta y)$.

For simplicity, we just choose $\Delta x = \Delta y$

$$u_{j,k} = \frac{1}{4} (u_{j+1,k} + u_{j-1,k} + u_{j,k+1} + u_{j,k-1})$$

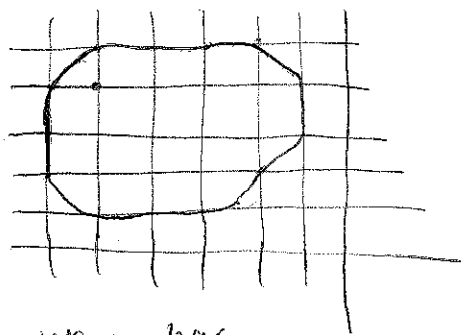
which is the average of the values at the four neighboring sites

$$\begin{array}{c} \cdot \frac{1}{4} \\ \frac{1}{4} \cdot \quad * \quad \cdot \frac{1}{4} \\ \cdot \frac{1}{4} \end{array}$$

This scheme has several nice properties:

- ① mean value property, the exact analog of the same property for the Laplace equation
- ② maximum principle in the discrete case

We have a discrete domain and no marching method is available in contrast to time-dependent problems.



Remark: the system we get in this way has exactly one solution. This can be shown by using the maximum principle in the discrete version.

In order to get the solution, we need to solve a system of linear equations $A\vec{u} = \vec{f}$

Jacobi method:

In matrix form: $A = D - L - U$

where D is the diagonal part of A , $-L$ and $-U$ are the strictly lower and upper triangular part of A , respectively.

$$A\vec{u} = \vec{f} \rightarrow D\vec{u} = (L+U)\vec{u} + \vec{f}$$
$$\rightarrow \vec{u} = D^{-1}(L+U)\vec{u} + \vec{f}$$

The Jacobi iteration works as follow

$$\vec{u}^{(n+1)} = D^{-1}(L+U)\vec{u}^{(n)} + \vec{f}$$

In comment form:

$$u_{j,k}^{(n+1)} = \frac{1}{4} (u_{j+1,k}^{(n)} + u_{j-1,k}^{(n)} + u_{j,k+1}^{(n)} + u_{j,k-1}^{(n)})$$

It can be shown $\lim_{n \rightarrow \infty} u_{j,k}^{(n)} = u_{j,k}$

This iteration is exactly the same calculation as if one were solving the two-dimensional heat equation

$$v_t = v_{xx} + v_{yy}$$

using centered differences for v_{xx} , v_{yy} and using the forward difference for v_t , with $\Delta x = \Delta y$

$$\text{and } \Delta t = \frac{(\Delta x)^2}{4}$$

In effect, we are solving the Dirichlet problem by taking the limit of the discretized $v(x,t)$ as $t \rightarrow \infty$.