## § 8.3 Approximations of waves

For the wave equation $u_{tt} = c^2 u_{xx}$, the simplest scheme is the one using centered differences for both terms

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{(\Delta t)^2} = c^2 \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}$$

or

$$u_j^{n+1} = s(u_{j+1}^n + u_{j-1}^n) + 2(1-s)u_j^n - u_j^{n-1}$$

where $s = c^2 \left(\frac{\Delta t}{\Delta x}\right)^2$

This is an explicit scheme since $u_j^{n+1}$ uses values at two previous time steps. Therefore, the first two rows $u_j^0$ and $u_j^1$ must be given as initial conditions.

Its template is

$$\begin{array}{cccc} n+1 & & * & \\ n & s & 2-2s & s \\ n-1 & & -1 & \end{array}$$

To kick start this marching scheme, we need the values at two initial time steps, for which we take

0 0 0 1 2 1 0 0 0

0 0 0 1 2 1 0 0 0

$s=2$ $\qquad u_j^{n+1} = 2(u_{j+1}^n - u_j^n + u_{j-1}^n) - u_j^{n-1}$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 8 | -12 | 4 | 13 | -22 | 13 | 4 | -12 | 8 |
| 0 | 4 | -2 | -3 | 6 | -3 | -2 | 4 | 0 |
| 0 | 0 | 2 | 1 | -2 | 1 | 2 | 0 | 0 |
| 0 | 0 | 0 | 1 | 2 | 1 | 0 | 0 | 0 |
| 0 | 0 | 0 | 1 | 2 | 1 | 0 | 0 | 0 |

$$\begin{array}{ccc} & * & \\ 2 & -2 & 2 \\ & -1 & \end{array}$$

Since $u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)]$ is the solution of $u_{tt} = c^2 u_{xx}$, you may find that the numerical solution is horrendous.

$s=1$ $\qquad u_j^{n+1} = u_{j+1}^n + u_{j-1}^n - u_j^{n-1}$

$$\begin{array}{ccc} & * & \\ 1 & 0 & 1 \\ & -1 & \end{array}$$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 |
| 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 |
| 0 | 0 | 0 | 1 | 2 | 1 | 0 | 0 | 0 |
| 0 | 0 | 0 | 1 | 2 | 1 | 0 | 0 | 0 |

This numerical solution is in good agreement with the true solution.

* Initial conditions $u(x,0) = \phi(x)$  $\frac{\partial u}{\partial t}(x,0) = \psi(x)$

$$u_j^0 = \phi(j\Delta x) \qquad \frac{u_j^1 - u_j^{-1}}{2\Delta t} = \psi(j\Delta x)$$

This approximation is chosen to have a $O(\Delta x^2)$ local truncation error in order to match $O(\Delta x^2) + O(\Delta t^2)$ truncation error of the scheme. Choosing $n=0$ in the scheme, we have

$$u_j^1 + u_j^{-1} = s(u_{j+1}^0 + u_{j-1}^0) + 2(1-s)\,u_j^0$$

Together with boundary conditions, this gives us the starting values

$$u_j^0 = \phi_j$$

$$u_j^1 = \frac{s}{2}(\phi_{j+1} + \phi_{j-1}) + (1-s)\phi_j + \psi_j \Delta t$$

the first two rows of the computation. Then we march ahead in time to get $u_j^2$, $u_j^3$, and so forth.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | 0 | 0 | 0 | 0 | 0 | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | |
| 0 | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | 0 | 0 | 0 | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | 0 | |
| 0 | 0 | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | 0 | 0 | |
| 0 | 0 | 0 | $\frac{1}{2}$ | 1 | 1 | 1 | $\frac{1}{2}$ | 0 | 0 | 0 | |
| 0 | 0 | 0 | 0 | 1 | 2 | 1 | 0 | 0 | 0 | 0 | $\phi(x)$ |

$\psi(x) \equiv 0$

*

+1  0  +1

−1

which gives a better approximation to the true solution.

* Stability Criterion

Assuming $u_j^n = e^{ikj\Delta x}\xi^n$, we have

$$\xi - 2 + \frac{1}{\xi} = s\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)$$

let $p = s(\cos k\Delta x - 1) + 1$, ② $\xi^2 - 2p\xi + 1 = 0$

$\xi = p \pm \sqrt{p^2-1}$ if the discriminant is positive, i.e. $p^2 > 1$

Since $p$ cannot be greater 1, $p < -1$, then

$$\xi_- = p - \sqrt{p^2-1} < p < -1$$

which would lead to instability

Therefore, we have to consider $p^2 \le 1$ case.

$$\xi = p \pm i\sqrt{1-p^2} \quad \text{with } |\xi| = 1$$

This is consistent with our intuition about wave equations, since $T_n = \xi^n = \cos n\theta + i \sin n\theta$

So the stability condition is equivalent to the requirement that the discriminant is nonpositive. $-1 \le p \le 1$, or

$$-1 \le 1 + s(\cos k\Delta x - 1) \le 1$$

The right inequality holds for $s > 0$

$$s \le \frac{2}{1-\cos k\Delta x}$$

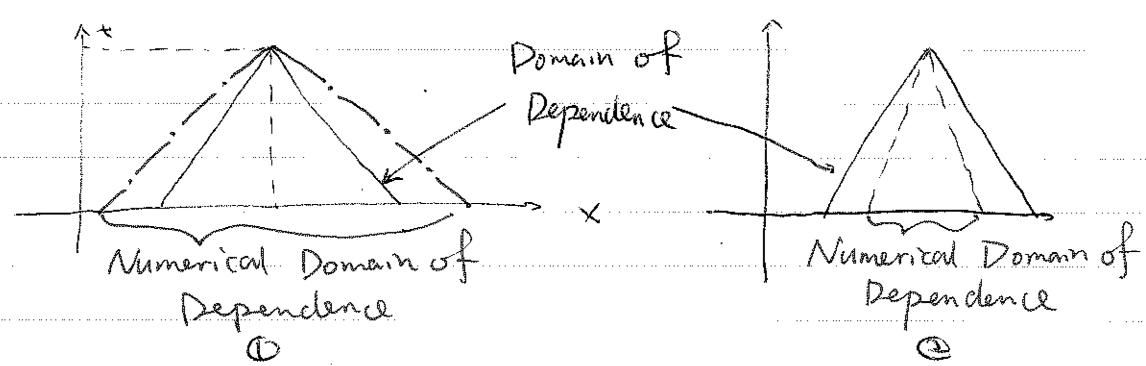
The worst case is $\cos k\Delta x \approx -1$ $\quad s \le \frac{2}{1-(-1)} = 1$

$$s = c^2\left(\frac{\Delta t}{\Delta x}\right)^2 \le 1 \rightarrow \frac{\Delta x}{\Delta t} \ge c$$

so the stability condition implies

speed of the scheme $\ge c$

Another way of understanding this stability is to compare the domains of dependence of the exact equation and the numerical scheme.




For ①, the scheme is stable, which the scheme in ② is unstable