§7.4 Green's Function in a half-space and a sphere

The Half-Space:

$\vec{x} = (x, y, z) \qquad D = \{z > 0\} \qquad \vec{x} \in D$

Reflected point $\vec{x}^* = (x, y, -z) \notin D$

We already know $-\frac{1}{4\pi|\vec{x}-\vec{x}_0|}$ satisfies two of the three conditions (i) & (iii) - required of the Green's function. We want to modify it to get (ii) as well.

We assert the Green's function for D is

$$G(\vec{x}, \vec{x}_0) = -\frac{1}{4\pi|\vec{x}-\vec{x}_0|} + \frac{1}{4\pi|\vec{x}-\vec{x}_0^*|} \qquad (*)$$

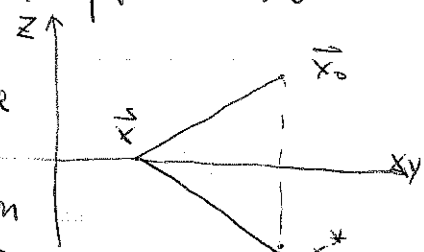
In coordinates.

$$G(\vec{x}, \vec{x}_0) = -\frac{1}{4\pi}\left[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right]^{-\frac{1}{2}} + \frac{1}{4\pi}\left[(x-x_0)^2 + (y-y_0)^2 + (z+z_0)^2\right]^{-\frac{1}{2}}$$

Now let's verify (*) by checking each of the three properties of G.

(i) Cleary, G is finite and differentiable except at $\vec{x}_0$.
Also $\Delta G = 0$

(ii) Let $\vec{x} \in \partial D \quad z = 0$. From the figure
$|\vec{x} - \vec{x}_0| = |\vec{x} - \vec{x}_0^*|$. Thus $G(\vec{x}, \vec{x}_0) = 0$



(iii) Since $\vec{x}_0^*$ is outside D, the function $-\frac{1}{4\pi|\vec{x}-\vec{x}_0^*|}$ has no singularity inside D,
so $G(\vec{x}, \vec{x}_0)$ has proper singularity at $\vec{x}_0$.

$$\begin{cases} \Delta u = 0 & \text{for } z > 0 \\ u(x, y, 0) = h(x, y) \end{cases}$$

We use the representation formula. Notice that

$$\frac{\partial G}{\partial n} = -\frac{\partial G}{\partial z}\Big|_{z=0} \quad \text{since } \vec{n} \text{ points downward}$$

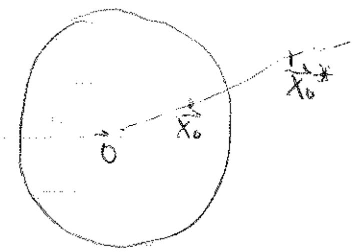
$$-\frac{\partial G}{\partial z} = \frac{1}{4\pi}\left(\frac{z+z_0}{|\vec{x}-\vec{x}_0^*|^3} - \frac{z-z_0}{|\vec{x}-\vec{x}_0|^3}\right) = \frac{1}{2\pi}\frac{z_0}{|\vec{x}-\vec{x}_0|^3} \quad \text{on } z = 0$$

then

$$u(\vec{x}_0) = \frac{z_0}{2\pi}\iint_{\partial D} \frac{h(\vec{x})}{|\vec{x}-\vec{x}_0|^3}\, dS$$

Or $u(x_0, y_0, z_0) = \frac{z_0}{2\pi} \iint [(x-x_0)^2 + (y-y_0)^2 + (z_0)^2]^{-\frac{3}{2}} h(x,y)\,dx\,dy$

in coordinates.



The Sphere:

$D = \{ |\vec{x}| < a \}$

Fix any nonzero point $\vec{x_0}$ in the ball

The reflected point $\vec{x_0}^*$ is defined by two properties.

(I) It is collinear with the origin $\vec{0}$ and the point $\vec{x_0}$

(II) Its distance from $\vec{0}$ is determined by $|\vec{x_0}|\,|\vec{x_0}^*| = a^2$

$$\vec{x_0}^* = a^2 \frac{\vec{x_0}}{|\vec{x_0}|^2}$$

Let $\vec{x}$ be any point, denote $|\vec{x} - \vec{x_0}| = \rho$, $|\vec{x} - \vec{x_0}^*| = \rho^*$, then the Green's function of the ball is

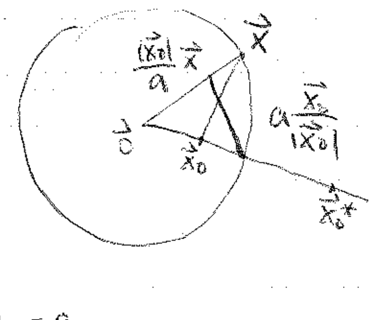
$$G(\vec{x}, \vec{x_0}) = -\frac{1}{4\pi\rho} + \frac{a}{|\vec{x_0}|} \frac{1}{4\pi\rho^*} \quad \text{if } \vec{x_0} \neq \vec{0}$$

Check three properties of Green's functions.

(i) $G(\vec{x}, \vec{x_0})$ has continuous second derivatives and is harmonic in $D \setminus \{\vec{x_0}\}$

(ii) $G(\vec{x}, \vec{x_0}) = 0$ if $\vec{x} \in \partial D$

From the congruent triangles



$$\left| \frac{|\vec{x_0}|}{a}\vec{x} - \frac{a}{|\vec{x_0}|}\vec{x_0} \right| = |\vec{x} - \vec{x_0}| = \rho$$

$$LHS = \frac{|\vec{x_0}|}{a} \left| \vec{x} - \frac{a^2}{|\vec{x_0}|^2}\vec{x_0} \right| = \frac{|\vec{x_0}|}{a} |\vec{x} - \vec{x_0}^*|$$

$$= \frac{r_0}{a}\rho^*$$

Thus $\rho = \frac{|\vec{x_0}|}{a}\rho^*$ for any $|\vec{x}| = a$

$G(\vec{x}, \vec{x_0}) = 0$ for any $|\vec{x}| = a$

If $\vec{x_0} = \vec{0}$ $\quad G(\vec{x}, \vec{0}) = -\frac{1}{4\pi|\vec{x}|} + \frac{1}{4\pi a}$

$G(\vec{x}, \vec{0}) = 0$ if $|\vec{x}| = a$

(iii) $G(\vec{x},\vec{x_0}) - \left(-\frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{x_0}|}\right) = \frac{a}{|\vec{x_0}|}\frac{1}{4\pi\rho^*}$

$= \frac{1}{4\pi}\frac{1}{\left|\frac{|\vec{x_0}|}{a}\vec{x} - \frac{a}{|\vec{x_0}|}\vec{x_0}\right|}$ is harmonic in D if $\vec{x_0} \neq \vec{0}$

$G(\vec{x},\vec{x_0}) - \left(-\frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{x_0}|}\right) = \frac{1}{4\pi a}$ is harmonic in D if $\vec{x_0} = \vec{0}$

Now let's write the solution of $\begin{cases} \Delta u = 0 & \text{in } |\vec{x}| < a \\ u = h & \text{on } |\vec{x}| = a \end{cases}$

$$u(\vec{x_0}) = \iint_{|x|=a} h(\vec{x})\frac{\partial}{\partial n}G(\vec{x},\vec{x_0})\,dS$$

Since $\rho^2 = |\vec{x}-\vec{x_0}|^2$, we have $\nabla\rho = \frac{\vec{x}-\vec{x_0}}{\rho}$

$\nabla\rho^* = \frac{\vec{x}-\vec{x_0}^*}{\rho^*}$, so

$$\nabla G = \frac{\vec{x}-\vec{x_0}}{4\pi\rho^3} - \frac{a}{|\vec{x_0}|}\frac{\vec{x}-\vec{x_0}^*}{4\pi\rho^{*3}}$$

Using $\vec{x_0}^* = \frac{a^2}{|\vec{x_0}|^2}\vec{x_0}$ and $\rho^* = \frac{a}{|\vec{x_0}|}\rho$ if $|\vec{x}| = a$

$$\nabla G = \frac{1}{4\pi\rho^3}\left[\vec{x}-\vec{x_0} - \left(\frac{|\vec{x_0}|}{a}\right)^2\vec{x} + \vec{x_0}\right]$$

and

$$\frac{\partial G}{\partial n} = \nabla G\cdot\frac{\vec{x}}{a} = \frac{a^2-|\vec{x_0}|^2}{4\pi a\rho^3}$$

then $$u(\vec{x_0}) = \frac{a^2-|\vec{x_0}|^2}{4\pi a}\iint_{|x|=a}\frac{h(\vec{x})}{|\vec{x}-\vec{x_0}|^3}\,dS$$

Or in spherical coordinates,

$$u(r_0,\theta_0,\phi_0) = \frac{a(a^2-r_0^2)}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{h(\theta,\phi)}{(a^2+r_0^2-2ar_0\cos\psi)^{\frac{3}{2}}}\sin\theta\,d\theta\,d\phi$$

where ψ is the angle between $\vec{x}$ and $\vec{x_0}$

In 2D $u_{xx}+u_{yy}=0$ in $x^2+y^2<a^2$ $u=h$ on $x^2+y^2=a^2$

$$G(\vec{x},\vec{x_0}) = \frac{1}{2\pi}\log\rho - \frac{1}{2\pi}\log\left(\frac{|\vec{x_0}|}{a}\rho^*\right)$$

and

$$u(\vec{x_0}) = \frac{a^2-|\vec{x_0}|^2}{2\pi a}\int_{|\vec{x}|=a}\frac{h(\vec{x})}{|\vec{x}-\vec{x_0}|^2}\,ds$$ $\leftarrow$ Poisson's Formula