§7.3 Green's Functions

Representation formula is not very useful since both u & $\frac{\partial u}{\partial n}$ on the boundary are required.

Def'n. $G(\vec{x})$ is called a Green's function for the operator $-\Delta$ in (the three dimensional) domain D at the point $\vec{x_0} \in D$ if it satisfies the following properties

(i) $G(\vec{x})$ has continuous second derivatives and is harmonic in $D \setminus \{\vec{x_0}\}$

(ii) $G(\vec{x}) = 0$ on the boundary of D.

(iii) $G(\vec{x}) - V(\vec{x})$ is finite at $\vec{x_0}$ and is harmonic in all of D.

$V(\vec{x}) = -\frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{x_0}|}$ in 3D & $V(\vec{x}) = \frac{1}{2\pi}\log|\vec{x}-\vec{x_0}|$ in 2D

From the definition, $G(\vec{x},\vec{x_0})$ satisfies

$$\begin{cases} \Delta G(\vec{x},\vec{x_0}) = \delta(\vec{x}-\vec{x_0}) & \text{in } D \\ G(\vec{x},\vec{x_0}) = 0 & \text{on } \partial D \end{cases}$$

Remark: $G(\vec{x},\vec{x_0})$ is unique for the Dirichlet problem.

Theorem: If $G(\vec{x},\vec{x_0})$ is a Green's function in D, then the solution of $\begin{cases} \Delta u = 0 \text{ in } D \\ u = h(x) \text{ on } \partial D \end{cases}$ in D is given by

$$u(\vec{x_0}) = \iint_{\partial D} u(\vec{x}) \frac{\partial G(\vec{x},\vec{x_0})}{\partial n} dS \qquad \forall \vec{x_0} \in D$$

Proof: Using (G2), ~~to the~~ pair (u, G), we have

$$\iiint_D (u\Delta G - G\Delta u)\, d\vec{x} = \iint_{\partial D} \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right) dS$$

$$RHS = \iint_{\partial D} u\frac{\partial G}{\partial n} dS \qquad LHS = \iiint_D u\Delta G\, d\vec{x} = \iiint_D u\,\delta(\vec{x}-\vec{x_0})\, d\vec{x}$$

$$= u(\vec{x_0})$$

$$u(\vec{x_0}) = \iint_{\partial D} u(\vec{x}) \frac{\partial G(\vec{x},\vec{x_0})}{\partial n} dS \qquad \#$$

Theorem: If $G(\vec{x},\vec{x_0})$ is a Green's function in D, then the solution to Dirichlet's problem for Poisson's equation $\Delta u = f(\vec{x})$ is given by $u(\vec{x}_0) = \iint_{\partial D} u(\vec{x}) \frac{\partial G(\vec{x}, \vec{x}_0)}{\partial n} dS + \iiint_D f(\vec{x}) G(\vec{x}, \vec{x}_0) d\vec{x}$

To find a solution formula for the Neumann problem, (ii) in the definition of the Green's function is replaced by

(ii_N) $\frac{\partial G}{\partial n} = c$ on the boundary of D for a suitable constant c.

$$\begin{cases} \Delta G(\vec{x}, \vec{x}_0) = \delta(\vec{x} - \vec{x}_0) \text{ in } D \\ \frac{\partial G(\vec{x}, \vec{x}_0)}{\partial n} = c \quad \text{on } \partial D \end{cases}$$

The divergence theorem then implies that

$$\iiint_D \Delta G(\vec{x}, \vec{x}_0) \, d\vec{x} = \iint_{\partial D} \frac{\partial G}{\partial n} dS$$

$LHS = \iiint_D \delta(\vec{x} - \vec{x}_0) \, d\vec{x} = 1$ $\quad RHS = \iint_{\partial D} c \, dS = c \, |\partial D|$

Hence $c = \frac{1}{|\partial D|}$

The solution formula for the Neuman problem

$$\begin{cases} \Delta u = 0 & \text{in } D \\ \frac{\partial u}{\partial n} = h(\vec{x}) & \text{on } \partial D \end{cases}$$

is given by

$$u(\vec{x}_0) = C - \iint_{\partial D} G(\vec{x}, \vec{x}_0) \frac{\partial u(\vec{x})}{\partial n} d\vec{x}, \quad C \text{ arbitrary constant}$$

$$\begin{cases} \Delta u = f & \text{in } D \\ \frac{\partial u}{\partial n} = h(\vec{x}) & \text{on } \partial D \end{cases}$$

the solution is then given by

$$u(\vec{x}_0) = C - \iint_D G(\vec{x}, \vec{x}_0) \frac{\partial u(\vec{x})}{\partial n} d\vec{x} + \iiint_D f(\vec{x}) G(\vec{x}, \vec{x}_0) d\vec{x}$$