

§ 7.2 Green's Second Identity

$$\iiint_D (u \Delta v - v \Delta u) d\vec{x} = \iint_{\partial D} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS \quad (G2)$$

~~valid~~ valid for any pair of functions u and v

Proof: From (G1) $\iint_{\partial D} v \frac{\partial u}{\partial n} dS = \iiint_D \nabla v \cdot \nabla u d\vec{x} + \iint_D v \Delta u d\vec{x}$

Applying (G1) for the pair v and u , we get

$$\iint_{\partial D} u \frac{\partial v}{\partial n} dS = \iiint_D \nabla u \cdot \nabla v d\vec{x} + \iint_D u \Delta v d\vec{x}$$

Subtracting, we get (G2)

Defn: A boundary condition is called symmetric for the operator Δ if the right side of (G2) vanishes for all pairs of functions u, v that satisfy the boundary condition.

Each of the three classical boundary conditions (Dirichlet, Neumann & Robin) is symmetric.

Representation Formulas: If $\Delta u = 0$ in D , then

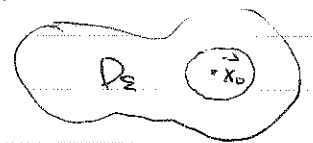
$$u(\vec{x}_0) = \iint_{\partial D} \left[-u(\vec{x}) \frac{\partial}{\partial n} \left(\frac{1}{|\vec{x} - \vec{x}_0|} \right) + \frac{1}{|\vec{x} - \vec{x}_0|} \frac{\partial u}{\partial n} \right] \frac{dS}{4\pi}$$

Proof: Substituting $v(\vec{x}) = -\frac{1}{4\pi|\vec{x} - \vec{x}_0|}$ into (G2), we have

$$\iiint_D (u \Delta v - v \Delta u) d\vec{x} = \iint_{\partial D} \left[u \frac{\partial}{\partial n} \left(\frac{1}{4\pi|\vec{x} - \vec{x}_0|} \right) - \frac{1}{4\pi|\vec{x} - \vec{x}_0|} \frac{\partial u}{\partial n} \right] dS$$

Since $\Delta u = 0$ $\Delta \left(-\frac{1}{4\pi|\vec{x} - \vec{x}_0|} \right) = 0$ except $\vec{x} = \vec{x}_0$

Let D_ϵ be the region D with this ball (of radius ϵ and center \vec{x}_0) excised



$$\iint_{\partial D_\epsilon} \left[u \frac{\partial}{\partial n} \left(-\frac{1}{4\pi|\vec{x} - \vec{x}_0|} \right) - \frac{1}{4\pi|\vec{x} - \vec{x}_0|} \frac{\partial u}{\partial n} \right] dS = 0$$

For simplicity, we assume $\vec{x}_0 = \vec{0}$

①

$$\bullet \iint_{\partial D} \left[-u(\vec{x}) \frac{\partial}{\partial n} \left(\frac{1}{|\vec{x} - \vec{x}_0|} \right) + \frac{1}{|\vec{x} - \vec{x}_0|} \frac{\partial u}{\partial n} \right] \frac{dS}{4\pi} =$$

$$- \iint_{|\vec{x}|=r} \left[-u(\vec{x}) \frac{\partial}{\partial n} \left(\frac{1}{|\vec{x}|} \right) + \frac{1}{|\vec{x}|} \frac{\partial u}{\partial n} \right] \frac{dS}{4\pi}$$

On the sphere $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$

The RHS = $-\iint_{r=\epsilon} \left[u \frac{\partial}{\partial r} \left(\frac{1}{r} \right) + \frac{1}{r} \frac{\partial u}{\partial r} \right] \frac{dS}{4\pi}$

$$= \frac{1}{4\pi\epsilon^2} \iint_{r=\epsilon} u dS + \frac{1}{4\pi\epsilon} \iint_{r=\epsilon} \frac{\partial u}{\partial r} dS$$

$$= \bar{u} + \epsilon \left(\frac{\partial u}{\partial r} \right)$$

where \bar{u} denotes the average value of $u(\vec{x})$ on the sphere $|\vec{x}| = r = \epsilon$

As $\epsilon \rightarrow 0$

$$\text{RHS} = u(\vec{0}) + 0 \frac{\partial u}{\partial r}(\vec{0}) = u(\vec{0})$$

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Remark:

① Representation formula in 2D

$$u(\vec{x}_0) = \frac{1}{2\pi} \iint_{\partial D} \left[u(\vec{x}) \frac{\partial}{\partial n} (\log |\vec{x} - \vec{x}_0|) - \frac{\partial u}{\partial n} \log |\vec{x} - \vec{x}_0| \right] dS$$

② Mean Value Property can be derived from representation formula by divergence theorem.

e.g., 2D $D = \{ |r| \leq a \}$ $\Delta u = 0$

$$u(\vec{x}_0) = \frac{1}{2\pi} \iint_{\partial D} \left[u(\vec{x}) \frac{\partial}{\partial n} (\log r) - \frac{\partial u}{\partial n} \log r \right] dS$$

$$= \frac{1}{2\pi a} \int_{\partial D} u(\vec{x}) dS - \frac{\log a}{2\pi} \int_{\partial D} \frac{\partial u}{\partial n} dS$$

The second term of RHS equals

$$- \frac{\log a}{2\pi} \int_{\partial D} \nabla u \cdot n dS = - \frac{\log a}{2\pi} \iint_D \nabla \cdot \nabla u d\vec{x} = 0$$

The first term of RHS is the average of $u(\vec{x})$ on the circumference.

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Let $v(\vec{x}) = -\frac{1}{4\pi|\vec{x} - \vec{x}_0|}$, we have $\Delta v = 0$ in $D \setminus B_\varepsilon$, where B_ε is a sphere with radius ε and \vec{x}_0 as the center.

$$\iiint_{D \setminus B_\varepsilon} w \Delta v d\vec{x} = 0 \quad \text{where } w \text{ is an arbitrary function}$$

$$\iiint_D w \Delta v d\vec{x} = \iiint_{B_\varepsilon} w \Delta v d\vec{x} \stackrel{(G2)}{=} \iiint_{B_\varepsilon} v \Delta w d\vec{x} + \iint_{\partial B_\varepsilon} \left(w \frac{\partial v}{\partial n} - v \frac{\partial w}{\partial n} \right) dS \quad \forall \varepsilon > 0$$

$$= \lim_{\varepsilon \rightarrow 0} \left(\iiint_{B_\varepsilon} v \Delta w d\vec{x} + \iint_{\partial B_\varepsilon} \left(w \frac{\partial v}{\partial n} - v \frac{\partial w}{\partial n} \right) dS \right)$$

$$= \lim_{\varepsilon \rightarrow 0} \left(\varepsilon^2 \Delta \tilde{w} + \tilde{w} + \varepsilon \frac{\partial \tilde{w}}{\partial r} \right) = w(\vec{x}_0)$$

This shows for any function w , $\iiint_D w \Delta v d\vec{x} = w(\vec{x}_0)$

Therefore, $\Delta v = \delta(\vec{x} - \vec{x}_0)$ with δ being the Dirac delta function. v is called the fundamental solution of Laplace equation.