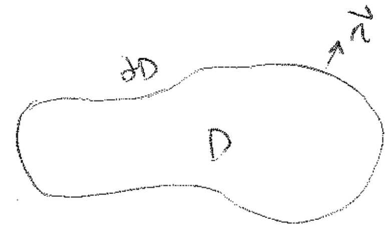
§7.1 Green's First Identity

The divergence theorem:



$$\iiint_D \operatorname{div}\vec{F}\, d\vec{x} = \iint_{\partial D} \vec{F}\cdot\vec{n}\, dS$$

where $\vec{F}$ is any vector function, D is a bounded solid region and $\vec{n}$ is the unit outer nomal on ∂D.

Starting from the product rule,

$$(v u_x)_x = v_x u_x + v u_{xx}$$

we sum x, y, z derivatives

$$(v u_x)_x + (v u_y)_y + (v u_z)_z = v_x u_x + v_y u_y + v_z u_z + v(u_{xx} + u_{yy} + u_{zz})$$

In vector form, $\nabla\cdot(v\nabla u) = \nabla v\cdot\nabla u + v\Delta u$.

We integrate and use the divergence theorem on the left side to get

$$\iint_{\partial D} v\frac{\partial u}{\partial n}\, dS = \iiint_D \nabla v\cdot\nabla u\, d\vec{x} + \iiint_D v\Delta u\, d\vec{x}$$

Green's first identity (G1)

If $v \equiv 1$, we have $\iint_{\partial D} \frac{\partial u}{\partial n}\, dS = \iiint_D \Delta u\, d\vec{x}$.

Consider the Neumann problem

$$\begin{cases} \Delta u = f(x) & \text{in } D \\ \frac{\partial u}{\partial n} = h(x) & \text{on } \partial D \end{cases} \qquad \iint_{\partial D} h\, dS = \iiint_D f\, d\vec{x}$$

$\leftarrow$ compatability condition

Mean value property: the average value of any harmonic function over a sphere equals its value at the center.

Proof: Without loss of generality $D = \{|\vec{x}| < a\}$ $\partial D = \{|\vec{x}| = a\}$

$$\vec{n} = \frac{\vec{x}}{|\vec{x}|} = \frac{\vec{x}}{r} \qquad \frac{\partial u}{\partial n} = \nabla u\cdot n = \frac{\partial u}{\partial x}\frac{x}{r} + \frac{\partial u}{\partial y}\frac{y}{r} + \frac{\partial u}{\partial z}\frac{z}{r} = \frac{\partial u}{\partial r}$$

$$\iint_{\partial D} \frac{\partial u}{\partial r}\, dS = 0$$

In spherical coordinates $\int_0^{2\pi}\int_0^{\pi} u_r(a,\theta,\phi)\, a^2 \sin\theta\, d\theta\, d\phi = 0$

Since the area of ∂D is $4\pi a^2$,

$$\frac{1}{4\pi a^2}\int_0^{2\pi}\int_0^{\pi} u_r(a,\theta,\phi)\, a^2\sin\theta\, d\theta\, d\phi = 0$$

$$\rightarrow \quad \frac{\partial}{\partial r}\left[\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u(r,\theta,\phi)\sin\theta\, d\theta\, d\phi\right]\Big|_{r=a} = 0 \quad \forall a>0$$

so $\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u(r,\theta,\phi)\sin\theta\, d\theta\, d\phi$ is independent of r.

Let $r\to 0$, we get

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u(\vec{0})\sin\theta\, d\theta\, d\phi = u(\vec{0})$$

That is

$$u(\vec{0}) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u(r,\theta,\phi)\sin\theta\, d\theta\, d\phi$$
$$= \frac{1}{|\partial D|}\iint_{\partial D} u\, ds$$

Remark: The maximum principle can be proved by the mean value property. Moreover, $\frac{\partial u}{\partial n} > 0$ at the maximum point, which is called Hopf maximum principle

Uniqueness of Dirichlet's Problem by Energy Method

Assume two harmonic functions with the same boundary data.

Define $u = u_1 - u_2$ harmonic and zero boundary data

From (G1), choosing $v = u$, we have

$$0 = \iint_{\partial D} u\frac{\partial u}{\partial n}\, ds = \iiint_D |\nabla u|^2\, d\vec{x}$$

By the first vanishing theorem it follows that $|\nabla u|^2 \equiv 0$ in D, then $u \equiv C$ in D. Since $u=0$ on ∂D, $u \equiv 0$ in D. This proves the uniqueness.

Uniqueness of Neumann's Problem: If $\Delta u = 0$ in D and $\frac{\partial u}{\partial n} = 0$ on ∂D, then u is a constant in D.

Dirichlet's Principle:

Define the energy as $E[w] = \frac{1}{2}\iiint_D |\nabla w|^2 \, d\vec{x}$

where $w(x)$ defined in D satisfies the Dirichlet BC

$$w = h(\vec{x}) \quad \text{on } \partial D$$

Let $u(\vec{x})$ be the unique harmonic function in D that satisfies $u = h(\vec{x})$ on ∂D. Let $w(\vec{x})$ be any function that satisfies $w = h(\vec{x})$ on ∂D. Then

$$E[w] \geq E[u]$$

u is called ground state in some physical contexts and $E[u]$ is the ground state energy.

Proof: Let $v = u - w$, then

$$E[w] = \frac{1}{2}\iiint_D |\nabla(u-v)|^2 \, d\vec{x}$$

$$= E[u] - \iiint_D \nabla u \cdot \nabla v \, d\vec{x} + E[v]$$

Since $\Delta u = 0$ in D, $v = 0$ on ∂D, by G1 we have

$$E[w] = E[u] + E[v] \geq E[u] \qquad \#$$