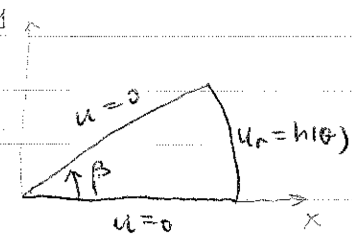
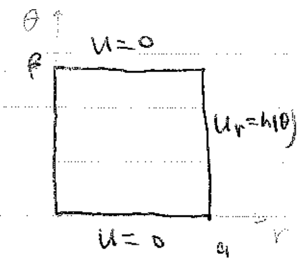
§ 6.4 Circles, Wedges & Annuli

Seperation of variables in polar coordinates works for domains whose boundaries are made up of concentric circles and rays.

Example: The wedge





$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \\ u(r,0) = u(r,\beta) = 0 \\ \frac{\partial u}{\partial r}(a,\theta) = h(\theta) \end{cases}$$

$u(r,\theta) = R(r)\,\Theta(\theta)$

$\rightarrow \quad \Theta'' + \lambda\Theta = 0 \qquad \Theta(0)=0 \qquad \Theta(\beta) = 0$

$\lambda_n = \left(\frac{n\pi}{\beta}\right)^2 \qquad \Theta_n(\theta) = \sin\frac{n\pi}{\beta}\theta \qquad n=1,2,\cdots$

and $\quad r^2R'' + rR' - \lambda R = 0 \qquad$ Euler-type

Try $R(r) = r^\alpha. \quad \alpha = \pm\frac{n\pi}{\beta}$

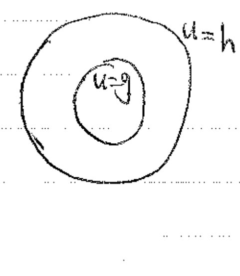
Since $r^{-\frac{n\pi}{\beta}}$ is infinite at the origin, we end up with the series $\quad u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{\frac{n\pi}{\beta}} \sin\frac{n\pi}{\beta}\theta$

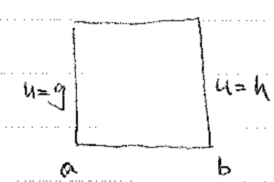
Using the inhomogeneous BC, underlying assumption: $h(\theta)$ is an odd function.

$$\boxed{h(\theta) = \sum_{n=1}^{\infty} A_n \frac{n\pi}{\beta} a^{\frac{n\pi}{\beta}-1} \sin\frac{n\pi}{\beta}\theta}$$

we have $\quad A_n = \frac{2}{n\pi} a^{1-\frac{n\pi}{\beta}} \int_0^\beta h(\theta) \sin\frac{n\pi}{\beta}\theta \, d\theta$

Example: The annulus





$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{in } \cancel{\alpha} a^2 < x^2+y^2 < b^2 \\ u = g(\theta) & \text{for } x^2+y^2 = a^2 \\ u = h(\theta) & \text{for } x^2+y^2 = b^2 \end{cases}$$

$u(r,\theta) = R(r)\,\Theta(\theta)$

$\rightarrow \quad \Theta'' + \lambda\Theta = 0 \qquad \Theta(\theta+2\pi) = \Theta(\theta)$

$\lambda_n = n^2 \qquad \Theta_n(\theta) = A\sin n\theta + B\cos n\theta \qquad n = 0,1,2,\cdots$

$r^2R'' + rR' - \lambda_n R = 0$

$n=0 \qquad R(r) = C + D\log r$

$n=1,2,\cdots \qquad R(r) = Cr^n + Dr^{-n}$

$$u(r,\theta) = \frac{1}{2}(C_0 + D_0\log r) + \sum_{n=1}^{\infty} (C_n r^n + D_n r^{-n})\cos n\theta + (A_n r^n + B_n r^{-n})\sin n\theta \quad (1)$$

Example: The exterior of a circle

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{for } x^2+y^2 > a^2 \\ u = h(\theta) & \text{for } x^2+y^2 = a^2 \\ u \text{ bounded as } x^2+y^2 \to \infty & \end{cases}$$

$$u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^{-n}(A_n \cos n\theta + B_n \sin n\theta)$$

and
$$h(\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} a^{-n}(A_n \cos n\theta + B_n \sin n\theta)$$

$$\to \quad A_n = \frac{a^n}{\pi}\int_0^{2\pi} h(\theta)\cos n\theta \, d\theta$$

$$B_n = \frac{a^n}{\pi}\int_0^{2\pi} h(\theta)\sin n\theta \, d\theta$$

This is the complete solution but it is one of the rare cases when the series can actually be summed.

$$u(r,\theta) = (r^2-a^2)\int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta-\phi) + r^2}\frac{d\phi}{2\pi} \quad \text{for } r>a$$