Chap. 6.3 Poisson's Formula

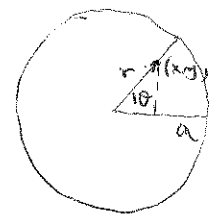
Consider the Dirichlet problem for a circle.

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{for } x^2+y^2 < a^2 \\ u = h(\theta) & \text{for } x^2+y^2 = a^2 \end{cases}$$

with radius a and boundary data h

Look for seperate solution in polar coordinates: $u = R(r)\,\Theta(\theta)$

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$$
$$= R''\Theta + \frac{1}{r} R'\Theta + \frac{1}{r^2} R\Theta'' = 0$$
$$r^2 \frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = 0$$



For $\Theta(\theta)$ we impose periodic BC: $\Theta(\theta+2\pi) = \Theta(\theta)$ $\forall \theta \in \mathbb{R}$

$\rightarrow$ $\lambda_n = n^2$ $\Theta(\theta) = A\cos n\theta + B\sin n\theta$ $n = 0, 1, 2, \cdots$

$$r^2 R'' + rR' - \lambda_n R = 0$$

$n=0$ $R(r) = A + B\log r$

$n = 1, 2, \cdots$ Euler-type equation try $R = r^\alpha$

$$\alpha(\alpha-1) r^\alpha + \alpha r^\alpha - n^2 r^\alpha = 0 \qquad \alpha = \pm n$$
$$R(r) = Cr^n + Dr^{-n}$$

At $r=0$ $\log r$, r^{-n} are infinite. We reject them

Summing the remaining solutions we have

$$u = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n (A_n\cos n\theta + B_n \sin n\theta)$$

Using inhomogeneous BC at $r=a$, we have

$$h(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta + B_n \sin n\theta)$$
$$A_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\cos n\phi \, d\phi \qquad n = 0, 1, \cdots$$
$$B_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\sin n\phi \, d\phi \qquad n = 1, 2, \cdots$$

Plugging A_n & B_n directly into u we get

$$u(r,\theta) = \int_0^{2\pi} h(\phi)\frac{d\phi}{2\pi} + \sum_{n=1}^{\infty} \frac{r^n}{\pi a^n}\int_0^{2\pi} h(\phi)\left\{\cos n\phi\cos n\theta + \sin n\phi \sin n\theta\right\} d\phi$$
$$= \int_0^{2\pi} h(\phi)\left\{1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta-\phi)\right\}\frac{d\phi}{2\pi}$$

(1)

Since $1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta-\phi) = 1+\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\theta-\phi)}$

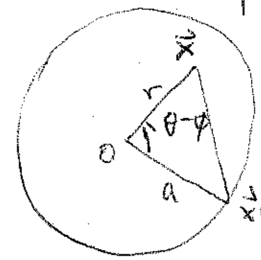
$+\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{-in(\theta-\phi)}$

$$= 1 + \frac{re^{i(\theta-\phi)}}{a-re^{i(\theta-\phi)}} + \frac{re^{-i(\theta-\phi)}}{a-re^{-i(\theta-\phi)}}$$

$$= \frac{a^2-r^2}{a^2-2ar\cos(\theta-\phi)+r^2},$$ we obtain

$$u(r,\theta) = (a^2-r^2)\int_0^{2\pi}\frac{h(\phi)}{a^2-2ar\cos(\theta-\phi)+r^2}\frac{d\phi}{2\pi} \quad \leftarrow \text{Poisson's Formula}$$

In a more geometric way,



$\vec{x}=(x,y) \Leftrightarrow (r,\theta)$

$\vec{x}' \longleftrightarrow (a,\phi)$

$|\vec{x}-\vec{x}'|^2 = a^2+r^2-2ar\cos(\theta-\phi)$

$$u(x) = \frac{a^2-|\vec{x}|^2}{2\pi a}\int_{|\vec{x}'|=a}\frac{u(\vec{x}')}{|\vec{x}-\vec{x}'|^2}\,ds'$$

The arc length element on the circumference is $ds' = a\,d\phi$

Thm:

Continuity: Let $h(\phi)=u(\vec{x}')$ be any continuous function on the circle $C=\partial D$. Then the Poisson formula provides the only harmonic function in D for which

$$\lim_{\vec{x}\to\vec{x}_0} u(\vec{x}) = h(\vec{x}_0) \quad \forall\ \vec{x}_0\in C$$

or $u(\vec{x})\in C(\bar{D})$ $\quad \bar{D}=D\cup C$

Differentiability: $u(\vec{x})$ possesses all partial derivatives of all orders in D i.e., $u(\vec{x})\in C^{\infty}(D)$

Mean Value Property:

Let u be a harmonic function in a disk D, continuous in its closure $\bar{D}$. Then the value of u at the center of D equals the average of u on its circumference

Proof: Choose coordinates with the origin 0 at the center of the circle. Put $\vec{x}=\vec{0}$ in Poisson's formula. Then

$$u(0) = \frac{a^2}{2\pi a}\int_{|x'|=a}\frac{u(\vec{x}')}{a^2}\,ds'$$

$= \frac{1}{2\pi a} \int_{|x'|=a} u(\vec{x}')\, ds'$, which is the average of u on the circumference $|x'| = a$

Remark: Mean value property can be used to show maximum principle of its strong form.