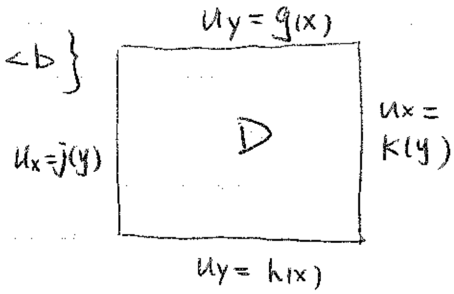
§6.2 Rectangles and Cubes

We demonstrate in this lecture how boundary values problems for Laplace's equation can be solved by separation of variables in the case of rectangles in 2D and cubes in 3D.

Example: D: rectangle = $\{0<x<a,\ 0<y<b\}$



$$\begin{cases} \Delta u = u_{xx} + u_{yy} = 0 & \text{in } D \\ u_y(x,0) = h(x), \quad u_y(x,b) = g(x), \\ u_x(0,y) = j(y), \quad u_x(a,y) = k(y). \end{cases}$$

Superposition principle: $u = u_1 + u_2 + u_3 + u_4$

u with boundary data (h, k, g, j)

u_1 with $(h, 0, 0, 0)$ $\quad u_2$ $(0, k, 0, 0)$

u_3 $(0, 0, g, 0)$ $\quad u_4$ $(0, 0, 0, j)$

We will choose u_4 as an example

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{in } D = (0,a)\times(0,b) \\ u_y(x,0) = 0, \quad u_y(x,b) = 0; & u_x(0,y) = j(y), \quad u_x(a,y) = 0 \end{cases}$$

and look for sperated solution $u(x,y) = X(x)Y(y)$

$$\rightarrow \quad X''Y + Y''X = 0 \quad \longrightarrow \quad \frac{X''}{X} + \frac{Y''}{Y} = 0$$

Let $\lambda = \frac{X''}{X}$, then $\frac{Y''}{Y} = -\lambda$ clearly λ is a constant independent of x, y.

$$X'' - \lambda X = 0$$
$$Y'' + \lambda Y = 0$$

with boundary condition

$$\left.\begin{array}{l} X(x)Y'(0) = 0 \\ X(x)Y'(b) = 0 \end{array}\right\} \longrightarrow \quad \begin{array}{l} Y'(0) = 0 \\ Y'(b) = 0 \end{array}$$

$$X'(a)Y(y) = 0 \quad \longrightarrow \quad X'(a) = 0$$

$$X'(0)Y(y) = j(y)$$

$$\begin{cases} Y'' + \lambda Y = 0 \\ Y'(0) = Y'(b) = 0 \end{cases} \longrightarrow \quad \begin{array}{l} \lambda_n = \left(\frac{n\pi}{b}\right)^2, \\ Y_n(y) = \cos\frac{n\pi y}{b} \end{array} \qquad n = 0, 1, 2, \dots$$

Then for

$$\lambda_0 = 0, \quad X_0'' = 0 \implies X_0(x) = \frac{A_0}{2} + \frac{B_0}{2}x$$

$$\lambda_n = \left(\frac{n\pi}{b}\right)^2, \ n = 1, 2, \dots \quad X_n'' - \left(\frac{n\pi}{b}\right)^2 X_n = 0 \implies X_n(x) = A_n e^{\frac{n\pi}{b}x} + B_n e^{-\frac{n\pi}{b}x}$$

$$X_n(x) = A_n \cosh \frac{n\pi x}{b} + B_n \sinh \frac{n\pi x}{b}$$

$$X'(a) = 0 \implies \begin{cases} B_0 = 0 \\ B_n = -A_n \tanh \frac{n\pi a}{b} \end{cases}$$

$$X_0(x)\, Y_0(y) = \frac{A_0}{2}$$

$$X_n(x)\, Y_n(y) = A_n \left(\cosh \frac{n\pi x}{b} - \tanh \frac{n\pi a}{b} \sinh \frac{n\pi x}{b} \right) \cos \frac{n\pi y}{b}$$

$$u(x, y) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \left(\cosh \frac{n\pi x}{b} - \tanh \frac{n\pi a}{b} \sinh \frac{n\pi x}{b} \right) \cos \frac{n\pi y}{b}$$

Using $u_x(0, y) = j(y)$, we have

$$\sum_{n=1}^{\infty} A_n \left(-\tanh \frac{n\pi a}{b} \right) \frac{n\pi}{b} \cos \frac{n\pi y}{b} = j(y)$$

and
$$A_n = \frac{2}{n\pi \left(-\tanh \frac{n\pi a}{b}\right)} \int_0^b j(y) \cos \frac{n\pi y}{b}\, dy$$

$$\longrightarrow u(x, y) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(\frac{2}{n\pi \left(-\tanh \frac{n\pi a}{b}\right)} \int_0^b j(y) \cos \frac{n\pi y}{b}\, dy \right) \left(\cosh \frac{n\pi x}{b} - \tanh \frac{n\pi a}{b} \sinh \frac{n\pi x}{b} \right) \cos \frac{n\pi y}{b}$$

The solution is determined up to a constant $\frac{A_0}{2}$, which is expected for a Neumann problem.

General strategy for solving Laplace equation in a rectangle and a cube:

I) Look for seperated solutions
II) Solve the eigenvalue problems with homogeneous b.c.
III) Solve other problems with obtained eigenvalues, and remaining homogeneous b.c
IV) Form the series solution, and find the coefficients from the inhomogeneous b.c.

Example: $u_{xx} + u_{yy} + u_{zz} = 0$ in $D = \{ 0 < x < \pi,\ 0 < y < \pi,\ 0 < z < \pi \}$

$$u(\pi, y, z) = g(y, z)$$

$$u(0, y, z) = u(x, 0, z) = u(x, \pi, z) = u(x, y, 0) = u(x, y, \pi) = 0$$

Look for seperated solution $u = X(x)\, Y(y)\, Z(z)$

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

$$X(0) = 0 \quad Y(0) = Y(\pi) = 0 \quad Z(0) = Z(\pi) = 0$$

$$Y(y) = \sin my \quad m = 1, 2, \cdots \quad Z(z) = \sin nz \quad n = 1, 2, \cdots$$

$$X'' - (m^2 + n^2) X = 0 \qquad X(0) = 0$$

$$X(x) = A \sinh(\sqrt{m^2+n^2}\, x)$$

Series solution

$$u(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sinh(\sqrt{m^2+n^2}\, x) \sin my \sin nz$$

and

$$g(y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sinh(\sqrt{m^2+n^2}\, \pi) \sin my \sin nz$$

$$\rightarrow \quad A_{mn} = \left(\int_0^{\pi} \int_0^{\pi} g(y, z) \sin my \sin nz \, dy \, dz \right) \frac{4}{\pi^2 \sinh(\sqrt{m^2+n^2}\, \pi)}$$